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Homomorphic images of semi-direct products

Lamies Joureus Nazzal

HOMOMORPHIC IMAGES OF SEMI-DIRECT PRODUCTS

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

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Approved by:

[redacted] 12/1/04

Zahid Hasan, Committee Chair, Date

[redacted]

Peter Williams, Committee Member

[redacted]

John Sarli, Committee Member

[redacted]

Peter Williams, Chair
Department of Mathematics

Terry Hallett

Terry Hallett,
Graduate Coordinator
Department of
Mathematics

ABSTRACT

Every finite non-abelian simple group is a homomorphic image of an infinite semi-direct product of the form $P = 2^{*n}:N$, where N is a subgroup of Sn, the Symmetric group of degree n. We will obtain several finite groups, in particular, various Projective General and Projective Special Linear groups as homomorphic images of $2^{*n}:N$. We will also demonstrate methods to construct computer-free proofs of existence of finite groups.

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I am grateful to my family, Dad and Mom, brothers and sisters with their families, for their continuous support and prayers. I am thankful to all my friends for their encouragement. Most of all, I extremely thank my Lord Jesus Christ for providing me with the privilege of accomplishing this thesis.

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CHAPTER ONE

BASIC DEFINITIONS AND NOTATIONS

Definitions:

G-Set. Let X be a set and G be a group, then X is a G-Set if there is a function $\alpha : G \times X \rightarrow X$, denoted by $\alpha\,(g, x) \mapsto gx$, such that:

i) $1x = x$ for all $x \in X$; and

ii) $g\,(hx) = (gh)\,x$ for all $g, h \in G$ and $x \in X$.

K-Transitive. Let X be a G-Set of degree n and let k ≤ n be a positive integer then X is K-transitive if, for every pair of k-tuples having distinct entries in X, say $(x_1, x_2, \ldots, x_k)$ and $(y_1, y_2, \ldots, y_k)$, there is $g \in G$ with $gx_i = y_i$ for all i in $\{1, 2, \ldots, k\}$.

Stabilizer of an element. If X is a G-Set and $x \in X$, then the stabilizer of x, denoted by G_x, is the subgroup

$$G_x = \{g \in G : gx = x\} \leq G$$

Faithful. A G-Set X with action α is faithful if $\tilde{\alpha} : G \rightarrow S_x$ is injective.

Conjugate. If $x \in G$, then a conjugate of x in G is an element of the form $a^{-1}xa$ for some $a \in G$.

Complement. Let K be a subgroup of a group G. Then a subgroup $Q \leq G$ is a complement of K in G if $K \cap Q = 1$ and $KQ = G$.

Semi-direct Product. A group G is a semi-direct product of K by Q, if K is normal subgroup of G and K has a complement $Q \equiv Q_1$.

Free Group. If X is a subset of a group F, then F is a free group with basis X if, for every group G and every function $f : X \to G$, there exists a unique homomorphism $\varphi : F \to G$ extending f.

Free Product. Let $\{A_i : i \in I\}$ be a family of groups. A free product of the A_i is a group P and a family of homomorphisms $j_i : A_i \to P$ such that, for every group G and every family of homomorphisms $f_i : A_i \to G$, there exists a unique homomorphism $\varphi : P \to G$ with $\varphi j_i = f_i$ for all i.

Word. A word on X is a sequence $w=(a_1, a_2, \dots)$, where $a_i \in X \cup X^{-1} \cup \{1\}$ for all i, such that all $a_i = 1$ from some point on; that is, there is an integer $n \geq 0$ with $a_i = 1$ for all $i \leq n$. In particular, the constant sequence $(1, 1, 1, \dots)$ is a word, called the empty word, and it is also denoted by 1.

Commutator. If $a, b \in G$, the commutator of a and b, denoted by $[a, b] = a\,b\,a^{-1}b^{-1}$.

Simple group. A group is simple if its only normal subgroups are the identity subgroup and the group itself.

The general linear group $GL_n(q)$ is the group of all linear automorphisms of an n-dimensional vector space over the field F_q, q being any prime power. The special linear group is the normal subgroup consisting of the automorphisms of determinant 1. The center of either of these groups consists of operations of the form $x \to kx$ (for $k \in F_q$), and we obtain the corresponding projective groups $PGL_n(q)$ and $PSL_n(q)$ by factoring out these centers.

The projective general linear group $PGL_2(q)$ consists of the linear maps given by:

$x \to \frac{ax+b}{cx+d}$; where a, b, c, d are elements of the field F_q, with $ad - bc \neq 0$, and $q = p^n$, where p is a prime.

The projective special linear group $PSL_2(q)$ is a subgroup of $PGL_2(q)$. $PSL_2(q)$ consists of the linear maps given by:

$x \to \frac{ax+b}{cx+d}$; where a, b, c, d are elements of the field F_q, with $ad - bc = 1$, or equivalently with $ad - bc$ any non-zero square in the field F_q.

$PSL_2(q)$ is generated by three operations

$\alpha : x \to x+1$

$\beta : x \to kx$

$\gamma : x \to -x^{-1}$

where k is an element of F_q whose powers are all squares.

Notation:

In this thesis permutations will be multiplied from left to right.

CHAPTER TWO

INTRODUCTION

Every finite non-abelian simple group is a homomorphic image of a progenitor of the form $P = 2^{*n}:N$, where N is a subgroup of S_n, the Symmetric group of degree n. (see Curtis[2]) and Curtis' simple proof is reproduced in this chapter. Symmetric presentations for almost all of the twenty six sporadic simple groups are known (see Curtis [7]) and these presentations lead to construction of most of these groups through manual double coset enumeration of the group over N.

The main purpose of this thesis is to describe methods of constructing computer-free proofs of existence of finite groups and give useful techniques to perform double coset enumeration of groups with symmetric presentations over their control groups. We explain two different methods for constructing these proofs.

First I am going to show Curtis' simple proof of a very important theorem.

Curtis' Theorem: Any finite non-abelian simple group is an image of a progenitor of form $P = 2^{*n} : N$, where N is a transitive subgroup of the symmetric group S_n.

Proof:

Let G be a finite non-abelian simple group. By Feit-Thompson Theorem, G has even order. (see Feit [3]).

It is not a 2-group, because if G is a 2-group then its center is non-trivial which contradicts the fact that G is a simple group.

Since the order of G is even, then G has a conjugacy class of elements of order 2, which generates a normal subgroup of G. But G is simple, so this conjugacy class of elements of order 2 generates G itself. Thus G is generated by its involutions. Now let M be a maximal subgroup of G. We can find an involution x of G that is not in M and then $<M,x> = G$, since M is maximal.

But the subgroup $< x^M >$ is normalized by M and x because $x^{-1}x^m x \in < x^M >$ for all $m \in M$, since $x \in < x^M >$

$\Rightarrow x^{-1} \in < x^M >$

$\Rightarrow x^{-1}x^m x \in < x^M >$

and so $< x^M >$ is normal in G.

By the simplicity of G, $< x^M > = G$ and so if $\left| x^M \right| = n$ then G is an image of $2^{*n}:M$, since there is a homomorphism

$\Phi : 2^{*n} : M \xrightarrow[Onto]{Hom} G$ defined by

$\Phi(x) = x \qquad \forall\, x \in 2^{*n}$

$\Phi(m) = m \qquad \forall\, m \in M$

It remains to show that M acts faithfully on the n elements of x^M by conjugation, that is M is a subgroup of S_n. But no non-identity element of M can commute with every element of the generating set x^M because the center of the non-abelian simple group G is 1. Thus, M acts faithfully on the elements of x^M by conjugation.

We note that G is an image of $2^{*n} : \mathrm{G}$, where $n = \left| x^G \right|$ and the set of symmetric generators is a complete conjugacy class of G. But the proof provides a richer source of pre-images than this.

Symmetric generation of a group.

Let G be a group and let $T=\{t_1, t_2, \ldots, t_n\} \subseteq G$. Define $\overline{T}=\{T_1, T_2, \ldots, T_n\}$, where $T_i=<t_i>$ for $i \in \{1,2,\ldots,n\}$ which is the cyclic subgroup of order m generated by t_i.

Let N be the control subgroup where $N=N_G(\overline{T})$.

T is a symmetric generating set for G iff

(i) $G=<T>$

(ii) N permutes $\overline{T}$ transitively.

Note that both i, and ii imply that G is a homomorphic image of the progenitor $m^{*n}:N$, which is an infinite semi-direct product of m^{*n} by N, where m^{*n} represents the free product of n copies of the cyclic group C_m extended by N, where m is the order of t_i. N will simply act as permutations of the n symmetric generators.

Any additional relation required to define G can be written in the form $\pi = w(t_1, t_2, \ldots, t_n)$, where $\pi \in N$ and w is a word in the symmetric generators. (see Curtis [10]).

In this thesis we consider the case when m = 2. Thus we seek homomorphic images of the semi-direct product $2^{*n}:N$, where 2^{*n} is the free product of n copies of the cyclic group C_2 of order 2, and N is a transitive permutation group on n letters. It is convenient to identify the n free generators and N with their respective images. Thus

$$G = \frac{2^{*n}:N}{\pi_1 w_1, \pi_2 w_2, \ldots, \pi_s w_s}$$
$$\cong \langle N, T \mid N_p, t_i^2 = 1, t_i^\pi = t_{(i)\pi}, \pi_1 w_1 = \pi_2 w_2 = \ldots = \pi_s w_s = 1 \rangle$$

where N_p represents the presentations of the control group N.

A presentation for $2^{*n}:N$ can be given by

$\langle x, y, t \mid \langle x, y \rangle \cong N, [t, N^0] = 1 = t^2 \rangle \cong 2^{*n}:N$,

where the set of relations includes a standard presentation for N in terms of generators x and y and N^0 denotes the point stabilizer of t_0, where t_0 is t ,in N in its action on n letters. (see Curtis [10]).

Double coset decomposition. Let G be a group defined by

$$G = \frac{2^{*n}:N}{\pi_1 w_1, \pi_2 w_2, \pi_3 w_3, \ldots}$$

We want to decompose it into double cosets of the form NxN; i.e. we wish to find a set $\{x_1, x_2, \ldots\}$ of elements of G such that

$G = Nx_1N \cup Nx_2N \cup Nx_3N \cup \ldots$

But we know that, for each i we can write x_i as $x_i = \pi_i w_i$, for some $\pi_i \in N$ and a word w_i in t_i, and hence the double coset decomposition will be

$G = N\pi_1 w_1 N \cup N\pi_2 w_2 N \cup N\pi_3 w_3 N \cup \ldots$

which can be simplified to

$G = Nw_1N \cup Nw_2N \cup Nw_3N \cup \ldots$ (since $\pi_i \in N \Rightarrow N\pi_i = N$)

$= N \cup Nw_2N \cup Nw_3N \cup \ldots$ (since w_1 is the identity)

The double cosets correspond to the orbits of N on the ordered k-tuples of letters which have no repetitions. And the number of single double cosets in each double coset is the number of distinct k-tuples in the orbit. When we factor by non-trivial relations, the number of single cosets contained in the double coset NwN becomes the index of the coset stabilizing subgroup $N^{(w)}$ in N, where

$N^{(w)} = \{\pi \in N \mid Nw\pi = N\}$

In other words, the number of single cosets in NwN is equal to $[N : N^{(w)}] = \frac{|N|}{|N^{(w)}|}$. (see Bray & Curtis [2]).

Cayley graph. The Cayley graph of a group $G=2^{*n}:N$ has as vertices the set of right cosets of N in G, i.e. the set of Nw_i, where w_i are words in t_i. The collapsed Cayley graph is the diagram in which each orbit of N in its action on the vertices by right multiplication is represented by a single node, labeled with the number of vertices that it contains. Lines between these nodes are labeled with integers to indicate how many edges from a vertex of one node lead to vertices of the other. (see Bray & Curtis [2]).

We give an easy example to demonstrate these ideas.

The group A_5 over S_3:

Consider the symmetric presentations of the progenitor $2^{*3}:S_3$ which given by:

$$<x, y, t \mid x^3, y^2, (y\,x)^2, t^2, (t, y)>$$

where the control group $N=S_3 \cong <x, y \mid x^3, y^2, (yx)^2>$,

x = (0 1 2),

y = (1 2),

and N^0, the stabilizer of 0 in N, is the group <y>.

We factor the progenitor by the following relations

$(xt_0)^5 = 1$, $(y^{x^{-1}}t_0)^5 = 1$ and $(y^{x^{-1}}t_0t_2)^3 = 1$

to obtain the group G.

Thus, $G \cong \dfrac{2^{*3}:S_3}{(xt_0)^5=1,\ (y^{x^{-1}}t_0)^5=1,\ (y^{x^{-1}}t_0t_2)^3=1}$

The index of N in G is 10. It is known that $G \cong A_5$.

Manual Double Coset enumeration of G over N:

The relation

$(xt_0)^5 = 1$

$\Rightarrow\ x\,t_0\,x\,t_0\,x\,t_0\,x\,t_0\,x\,t_0 = 1$

$\Rightarrow\ x^2\,x^{-1}\,t_0\,x\,t_0\,x^3\,x^{-2}\,t_0\,x^2\,x^{-1}\,t_0\,x\,t_0 = 1$

$\Rightarrow\ x^2\,t_0{}^{x}\,t_0\,t_0{}^{x^2}\,t_0{}^{x}\,t_0 = 1$

$\Rightarrow\ (0\ 2\ 1)\,t_1\,t_0\,t_2\,t_1\,t_0 = 1$

$\Rightarrow\ (0\ 2\ 1)\,t_1\,t_0 = t_0\,t_1\,t_2$ (1)

$\Rightarrow\ N\,t_1\,t_0 = N\,t_0\,t_1\,t_2$

Let $y^{x^{-1}} = \pi$. Then the relation $(y^{x^{-1}}t_0)^5 = 1$ becomes

$\pi\,t_0\,\pi\,t_0\,\pi\,t_0\,\pi\,t_0\,\pi\,t_0 = 1$

$\Rightarrow\ \pi\,t_0\,\pi^2\,\pi^{-1}\,t_0\,\pi\,t_0\,\pi^2\,\pi^{-1}\,t_0\,\pi\,t_0 = 1$

$\Rightarrow\ \pi\,t_0\,t_0{}^{\pi}\,t_0\,t_0{}^{\pi}\,t_0 = 1$

$\Rightarrow\ (0\ 1)\,t_0\,t_1\,t_0\,t_1\,t_0 = 1$

$\Rightarrow\ N\,t_0\,t_1 = N\,t_0\,t_1\,t_0$ (2)

The relation $(y^{x^{-1}}t_0t_2)^3 = 1$ becomes

$\pi\, t_0 t_2\, \pi\, t_0\, t_2\, \pi\, t_0\, t_2 = 1$

$\Rightarrow\ \pi\, t_0\, t_2\, (t_0\, t_2)^{\pi}\, t_0\, t_2 = 1$ (since $\pi = (0\ 1)$, so $\pi = \pi^{-1}$)

$\Rightarrow\ (0\ 1)\, t_0\, t_2\, t_1\, t_2\, t_0\, t_2 = 1$

$\Rightarrow\ N\, t_0\, t_2\, t_1 = N\, t_2\, t_0\, t_2$ (3)

Now $N\, t_0\, t_1 = N\, t_0\, t_1\, t_0$ (by 2) implies that [i j i]= [i j], since N is three transitive on T.

And $N\, t_1\, t_0 = N\, t_0\, t_1\, t_2$ (by 1) implies that [i j k]= [j i]

This indicates the double coset diagram:

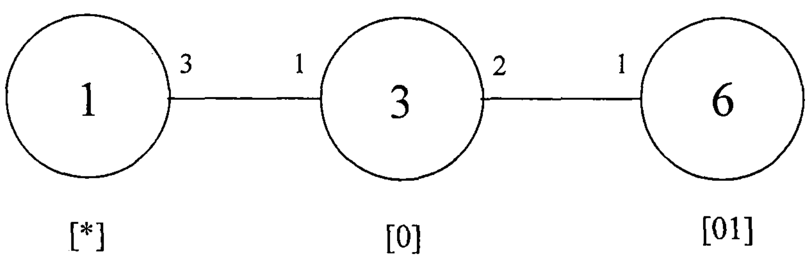

Figure 1. Cayley Graph of The Group A_5 Over S_3

Now we can calculate the action of the three symmetric generators on the cosets of this group. For example, in order to calculate this for the symmetric generator t_0, we start with the identity coset N or * and then multiply on the right by t_0, the result is Nt_0 or 0, where 0 stands for the coset Nt_0. Then repeat the process by multiplying again on the right by t_0, the result now is $Nt_0t_0 = N$. Now start with a new single coset and repeat the process. (Note that the permutation for the symmetric generators t_i's will be a product of two cycles).

We obtain:

t_0: (* 0)(1 10)(2 20)(01)(02)(12 21)

t_1: (* 1)(0 01)(2 21)(10)(20 02)(12)

t_2: (* 2)(0 02)(1 12)(01 10)(20)(21)

We relabel the cosets according to the following scheme

*	0	1	2	01	10	02	20	12	21
1	2	3	4	5	6	7	8	9	10

The three symmetric generators under this relabeling become:

$t_0 = (1\ 2)(3\ 6)(4\ 8)(9\ 10)$

$t_1 = (1\ 3)(2\ 5)(4\ 10)(8\ 7)$

$t_2 = (1\ 4)(2\ 7)(3\ 9)(5\ 6)$

We can now show the isomorphism, that is $G \cong A_5$:

We obtained the collapsed Cayley graph from the information contained in the following computations

$$\frac{|N|}{|N|}+\frac{|N|}{|N^{(0)}|}+\frac{|N|}{|N^{(01)}|}$$

$$=\frac{6}{6}+\frac{6}{2}+\frac{6}{1}$$

$$=1+3+6=10$$

this tells us that the maximum possible index of N in G is 10. It follows that the order of the image group G is at most $|N| * 10 = 6 * 10 = 60$. The order of G can be established by regarding G as a permutation group on the 10 cosets that we have found.

The action of the control group N on the cosets is

$x: (2, 3, 4)(5, 9, 8)(6, 10, 7)$

$y: (3, 4)(5, 7)(6, 8)(9, 10)$

The action of the symmetric generator t_0 is given by:

$t_0: (1, 2)(3, 6)(4, 8)(9, 10)$

The action of x and y, and hence N, on the symmetric generators is as follows:

$x: (t_0, t_1, t_2)$

$y: (t_1, t_2)$

We note that xy has order 2, and hence $N = \langle x, y\rangle \cong S_3$.

Now we check our relations; that is

$t_0^{t_1 t_0 t_2 t_1 t_0} = (1, 3)(2, 5)(4, 10)(8, 7) = t_1$

$t_1^{t_1 t_0 t_2 t_1 t_0} = (1, 4)(2, 7)(3, 9)(5, 6) = t_2$

$t_2^{t_1 t_0 t_2 t_1 t_0} = (1, 2)(3, 6)(4, 8)(9, 10) = t_0$

This means that $t_1 t_0 t_2 t_1 t_0$ acts as the permutation (0 1 2) on the symmetric generators, that is $t_1 t_0 t_2 t_1 t_0 = (0\ 1\ 2)$, which gives us the first relation.

Similarly,

$t_0^{t_0 t_1 t_0 t_1 t_0} = (1, 3)(2, 5)(4, 10)(8, 7) = t_1$

$t_1^{t_0 t_1 t_0 t_1 t_0} = (1, 2)(3, 6)(4, 8)(9, 10) = t_0$

$t_2^{t_0 t_1 t_0 t_1 t_0} = (1, 4)(2, 7)(3, 9)(5, 6) = t_2$

this means that $t_0 t_1 t_0 t_1 t_0$ acts as the permutation (0 1) on the symmetric generators, that is $t_0 t_1 t_0 t_1 t_0 = (0\ 1)$, and that proves our second relation.

Also,

$t_0^{t_0 t_2 t_1 t_2 t_0 t_2} = (1, 3)(2, 5)(4, 10)(8, 7) = t_1$

$t_1^{t_0 t_2 t_1 t_2 t_0 t_2} = (1, 2)(3, 6)(4, 8)(9, 10) = t_0$

$t_2^{t_0 t_2 t_1 t_2 t_0 t_2} = (1, 4)(2, 7)(3, 9)(5, 6) = t_2$

this means that $t_0 t_2 t_1 t_2 t_0 t_2$ acts as the permutation (0 1) on the symmetric generators, that is $t_0 t_2 t_1 t_2 t_0 t_2 = (0\ 1)$, and that gives the third relation.

We note that the elements x, y and t_0 generate the whole group, A_5, and so A_5 is the image of G. Thus, $|G| \geq |A_5|$.

We also have that $|G| \leq 60 = |A_5|$.

Therefore, $|G| \leq 60 = |A_5| \leq |G|$,

which proves the isomorphism, that is $G \cong A_5$.

In the next chapters we are going to apply these ideas to various finite groups including the groups PGL2(7), L2(11), PGL2(11), PSL2(23), and PGL2(17).

CHAPTER THREE

THE GROUP $PGL_2(7)$ OVER S_4

A symmetric presentations for the progenitor $2^{*4}: S_4$ is given by: $<x, y, t \mid x^6, y^2, (x\,y)^3, t^2, (t, y), (t^x, y)>$,

where the control group $N = S_4 \cong <x, y \mid x^3, y^2, (x\,y)^2>$

x = (0 1 2 3),

y = (2 3),

and $N^0 = <y, x\,y\,x^{-1}>$.

We factor the progenitor by the following relations

$(x\,t_0)^6 = 1$ and $(2\,3) = (t_0\,t_1)^2$ to obtain the group G,

$$G \cong \frac{2^{*4} : S_4}{(x\,t_0)^6 = 1, (2\,3) = (t_0\,t_1)^2}$$

The index of N in G is 14.

Manual Double Coset Enumeration

The relation $(x\,t_0)^6 = 1$

$\Rightarrow\; x\,t_0\,x\,t_0\,x\,t_0\,x\,t_0\,x\,t_0\,x\,t_0 = 1$

$\Rightarrow\; x^2\,x^{-1}\,t_0\,x\,t_0\,x^4\,x^{-3}\,t_0\,x^3\,x^{-2}\,t_0\,x^2\,x^{-1}\,t_0\,x\,t_0 = 1$

$\Rightarrow\; x^2\,t_0^{\,x}\,t_0\,t_0^{\,x^3}\,t_0^{\,x^2}\,t_0^{\,x}\,t_0 = 1$

$\Rightarrow \ (0\ 2)\,(1\ 3)\,t_1\ t_0\ t_3\ t_2\ t_1\ t_0 = 1$

$\Rightarrow \ (0\ 2)\,(1\ 3)\,t_1\ t_0\ t_3 = t_0\ t_1\ t_2 \qquad (1)$

$\Rightarrow \ N\,t_1\ t_0\ t_3 = N\,t_0\ t_1\ t_2$

The relation $(2\ 3) = (t_0\ t_1)^2$:

$(2\ 3) = (t_0\ t_1)^2 \Rightarrow \ (2\ 3) = t_0\ t_1\ t_0\ t_1$

$\Rightarrow \ (2\ 3)\ t_1\ t_0 = t_0\ t_1 \qquad (2)$

$\Rightarrow \ N\,t_1\ t_0 = N\,t_0\ t_1$

Therefore the double coset [i j] contains 6 single cosets since each coset has two names.

Also by (2), we have

$(2\ 3)\ t_1\ t_0\ t_1 = t_0$

$\Rightarrow$ [i j i] = [j] for all i and j in {0, 1, 2, 3}, since N is four transitive on T.

We now consider the double coset [i j k], where i ≠ j ≠ k

By (1) we have

$N\,t_1\ t_0\ t_3 = N\,t_0\ t_1\ t_2$

Similarly, by using (2), we can write

$N\,t_1\ t_0\ t_2 = N\,t_0\ t_1\ t_2$

$N\,t_1\ t_0\ t_3 = N\,t_0\ t_1\ t_3$

$N\,t_2\ t_3\ t_0 = N\,t_3\ t_2\ t_0$

$N t_2 t_3 t_1 = N t_3 t_2 t_1$

Thus, we can write

$t_1 t_0 t_2 = t_0 t_1 t_2 = t_1 t_0 t_3 = t_0 t_1 t_3 = t_2 t_3 t_0 = t_3 t_2 t_0 = t_2 t_3 t_1 = t_3 t_2 t_1$

Therefore the double coset [i j k] contains 3 single cosets

Since each coset has 8 names.

Thus, Cayley graph of G over N has the form:

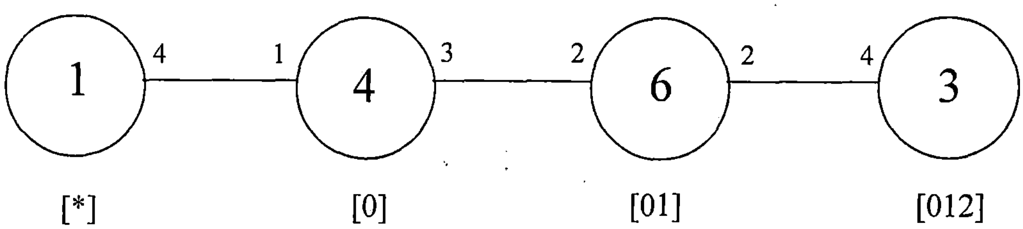

Figure 2. Cayley Graph of The Group $PGL_2(7)$ Over S_4

We obtain:

t_0: (* 0)(1 10)(2 20)(3 30)(12 120)(13 130)(23 230)

t_1: (* 1)(0 01)(2 21)(3 31)(20 201)(30 301)(23 231)

t_2: (* 2)(0 02)(1 12)(3 32)(10 102)(30 302)(31 312)

t_3: (* 3)(0 03)(1 13)(2 23)(01 013)(02 023)(12 123)

We rename the cosets as follows

*	0	1	2	3	01	02	03	12	13	23	012	021	210
1	2	3	4	5	6	7	8	9	10	11	12	13	14

Then, the four symmetric generators become:

t_0: (1 2)(3 6)(4 7)(5 8)(9 14)(10 13)(11 12)

t_1: (1 3)(2 6)(4 9)(5 10)(7 13)(8 14)(11 12)

t_2: (1 4)(2 7)(3 9)(5 11)(6 12)(8 14)(10 13)

t_3: (1 5)(2 8)(3 10)(4 11)(6 12)(7 13)(9 14)

Proof of Isomorphism

We obtained the collapsed Cayley graph from the information contained in the following computations

$$\frac{|N|}{|N|} + \frac{|N|}{|N^{(0)}|} + \frac{|N|}{|N^{(01)}|} + \frac{|N|}{|N^{(012)}|}$$

$$= \frac{24}{24} + \frac{24}{6} + \frac{24}{4} + \frac{24}{8}$$

$$= 1 + 4 + 6 + 3 = 14 .$$

Thus the maximum possible index of N in G is 14. It follows that the order of the image group G is at most $|N| * 14 = 24 * 14 = 336$. The order of G can be established by regarding G as a permutation group on the 10 cosets that we have found.

The action of the control group N on the cosets is

x: (2, 3, 4, 5) (6, 9, 11, 8) (7, 10) (12, 14)

y: (4, 5) (7, 8) (9, 10) (13, 14)

We note that xy has order 3, and hence $N = \langle x, y\rangle \cong S_4$.

The action of the symmetric generator t_0 on the cosets is given by:

$t_0 : (1, 2)\,(3, 6)\,(4, 7)\,(5, 8)\,(9, 14)\,(10, 13)\,(11, 12)$

Then the action of x and y, and hence N, on the symmetric generators is as follows:

$x : (t_0, t_1, t_2, t_3)$

$y : (t_2, t_3)$

We now check our relations; that is

$t_0^{t_0\, t_1\, t_2\, t_3\, t_0\, t_1} = (1, 4)\,(2, 7)\,(3, 9)\,(5, 11)\,(6, 12)\,(8, 14)\,(10, 13) = t_2$

$t_1^{t_0\, t_1\, t_2\, t_3\, t_0\, t_1} = (1, 5)\,(2, 8)\,(3, 10)\,(4, 11)\,(6, 12)\,(7, 13)\,(9, 14) = t_3$

$t_2^{t_0\, t_1\, t_2\, t_3\, t_0\, t_1} = (1, 2)\,(3, 6)\,(4, 7)\,(5, 8)\,(9, 14)\,(10, 13)\,(11, 12) = t_0$

$t_3^{t_0\, t_1\, t_2\, t_3\, t_0\, t_1} = (1, 3)\,(2, 6)\,(4, 9)\,(5, 10)\,(7, 13)\,(8, 14)\,(11, 12) = t_1$

This means that $t_0\, t_1\, t_2\, t_3\, t_0\, t_1$ acts as the permutation (0 2)(1 3) on the symmetric generators, that is $t_0\, t_1\, t_2\, t_3\, t_0\, t_1 = (0\,2)(1\,3)$, which gives is our relation (1).

Similarly,

$t_0^{(t_0\, t_1)^2} = (1, 2)\,(3, 6)\,(4, 7)\,(5, 8)\,(9, 14)\,(10, 13)\,(11, 12) = t_0$

$t_1^{(t_0\, t_1)^2} = (1, 3)\,(2, 6)\,(4, 9)\,(5, 10)\,(7, 13)\,(8, 14)\,(11, 12) = t_1$

$t_2^{(t_0\, t_1)^2} = (1, 5)\,(2, 8)\,(3, 10)\,(4, 11)\,(6, 12)\,(7, 13)\,(9, 14) = t_3$

$t_3^{(t_0\, t_1)^2} = (1, 4)\,(2, 7)\,(3, 9)\,(5, 11)\,(6, 12)\,(8, 14)\,(10, 13) = t_2$

Thus $(t_0\, t_1)^2$ acts as the transposition (2 3) on the symmetric generators, that is $(t_0\, t_1)^2 = (2\, 3)$, and that proves our relation (2).

Since S_4 is maximal in $PGL_2(7)$ and $t_0 \notin S_4$, x, y and t_0 generate the whole group, $PGL_2(7)$, and so $PGL_2(7)$ is a homomorphic image of G. Thus, $|G| \geq |PGL_2(7)|$.

We also have that $|G| \leq 336 = |PGL_2(7)|$.

Therefore, $|G| \leq 336 = |PGL_2(7)| \leq |G|$,

which proves the isomorphism, that is $G \cong PGL_2(7)$.

CHAPTER FOUR

SOME OTHER GROUPS

A Group G Over S_4

Symmetric presentations of the progenitor $2^{*4}:S_4$, can be written as: $<x, y, t \mid x^4, y^2, (x\,y)^3, t^2, (t, y), (t^x, y)>$,

where the control group $N = S_4 \cong <x, y \mid x^3, y^2, (x\,y)^2>$,

x = (0 1 2 3),

y = (2 3),

and $N^0 = <y, x\,y\,x^{-1}>$.

We factor the progenitor by the following relations

$t_0\,t_1 = t_1\,t_0$ to obtain G,

$$G \cong \frac{2^{*4}:S_4}{t_0\,t_1 = t_1\,t_0}$$

The index of N in G is 16.

<u>Manual Double Coset Enumeration</u>

The given relation is

$t_0\,t_1 = t_1\,t_0$ (1)

This means that the double coset [i j] contains 6 single cosets because every coset has two names.

Also by (1), we have

$t_0\, t_1\, t_0 = t_1$

$\Rightarrow$ [i j i] = [j] for all i and j in {0, 1, 2, 3}, since N is four transitive on T.

Now consider the double coset [i j k], where $i \neq j \neq k$

$t_0\, t_1\, t_2 = t_0\, t_2\, t_1 = t_1\, t_0\, t_2 = t_1\, t_2\, t_0 = t_2\, t_0\, t_1 = t_2\, t_1\, t_0$,

$t_0\, t_1\, t_3 = t_0\, t_3\, t_1 = t_1\, t_0\, t_3 = t_1\, t_3\, t_0 = t_3\, t_0\, t_1 = t_3\, t_1\, t_0$,

$t_0\, t_2\, t_3 = t_0\, t_3\, t_2 = t_2\, t_0\, t_3 = t_2\, t_3\, t_0 = t_3\, t_0\, t_2 = t_3\, t_2\, t_0$,

and $t_1\, t_2\, t_3 = t_1\, t_3\, t_2 = t_2\, t_1\, t_3 = t_2\, t_3\, t_1 = t_3\, t_1\, t_2 = t_3\, t_2\, t_1$.

(since $t_i\, t_j = t_j\, t_i$ for all i and j in {0, 1, 2, 3})

Therefore the double coset [i j k] contains 4 single cosets

Since each coset has 6 names.

Finally, the double coset [i j k l], where $i \neq j \neq k \neq l$, consists of a single coset, because

$t_0\, t_1\, t_2\, t_3 = t_0\, t_1\, t_3\, t_2 = t_0\, t_2\, t_3\, t_1 = t_1\, t_2\, t_3\, t_0$

(since $t_i\, t_j = t_j\, t_i$ for all i and j in {0, 1, 2, 3}).

Thus, Cayley graph of G over N has the form:

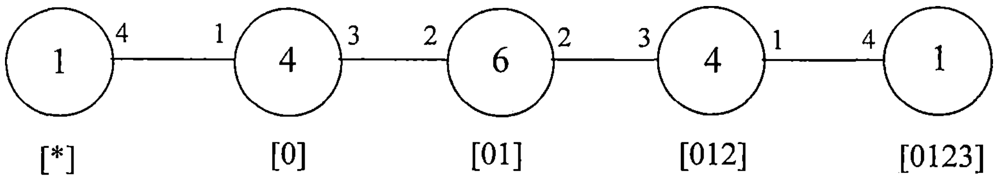

Figure 3. Cayley Graph of The Group G Over S_4

We obtain:

t_0:(* 0)(1 10)(2 20)(3 30)(12 120)(13 130)(23 230)(123 1230)

t_1:(* 1)(0 01)(2 21)(3 31)(02 021)(03 031)(23 231)(230 2301)

t_2:(* 2)(0 02)(1 12)(3 32)(01 012)(03 032)(13 132)(013 0132)

t_3:(* 3)(1 13)(2 23)(01 013)(02 023)(03 0)(12 123)(012 0123)

We rename the cosets according to the following scheme

*	0	1	2	3	01	02	03	12	13	23	012	013	023	123	0123
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

Then, the four symmetric generators become:

t_0: (1 2)(3 6)(4 7)(5 8)(9 12)(10 13)(11 14)(15 16)

t_1: (1 3)(2 6)(4 9)(5 10)(7 12)(8 13)(11 15)(14 16)

t_2: (1 4)(2 7)(3 9)(5 11)(6 12)(8 14)(10 15)(13 16)

t_3: (1 5)(3 10)(4 11)(6 13)(7 14)(8 2)(9 15)(12 16)

<u>Proof of Isomorphism</u>

We obtained the collapsed Cayley graph from the information contained in the following computations

$$\frac{|N|}{|N|}+\frac{|N|}{|N^{(0)}|}+\frac{|N|}{|N^{(01)}|}+\frac{|N|}{|N^{(012)}|}+\frac{|N|}{|N^{(0123)}|}$$

$$=\frac{24}{24}+\frac{24}{6}+\frac{24}{4}+\frac{24}{6}+\frac{24}{24}$$

$$=1+4+6+4+1=16$$

this tells us that the maximum possible index of N in G is 16. It follows that the order of the image group G is at most $|N| * 16 = 24 * 16 = 384$. The order of G can be confirmed by representing G as a permutation group on the 16 cosets that we have found.

The action of the control group N on the cosets is

x: (2, 3, 4, 5) (6, 9, 11, 8) (7, 10) (12, 15, 14, 13)

y: (4, 5) (7, 8) (9, 10) (12, 13)

We note that xy has order 3, and hence $N = \langle x, y\rangle \cong S_4$.

The action of the symmetric generator t_0 is:

t_0: (1, 2) (3, 6) (4, 7) (5, 8) (9, 12) (10, 13) (11, 14) (15, 16)

The action of x and y, and hence N, on the symmetric generators is as follows:

$x: (t_0, t_1, t_2, t_3)$

$y: (t_2, t_3)$

Now we check our relation; that is

$t_0^{(t_0 t_1)^2} = (1, 2)(3, 6)(4, 7)(5, 8)(9, 12)(10, 13)(11, 14)(15, 16) = t_0$,

$t_1^{(t_0 t_1)^2} = (1, 3)(2, 6)(4, 9)(5, 10)(7, 12)(8, 13)(11, 15)(14, 16) = t_1$,

$t_2^{(t_0 t_1)^2} = (1, 4)(3, 9)(5, 11)(6, 12)(7, 2)(8, 14)(10, 15)(13, 16) = t_2$,

$t_3^{(t_0 t_1)^2} = (1, 5)(3, 10)(4, 11)(6, 13)(7, 14)(8, 2)(9, 15)(12, 16) = t_3$.

Therefore, $(t_0 t_1)^2$ fixes all the symmetric generators, and it acts as the identity on t_0, t_1, t_2, t_3, that is $(t_0 t_1)^2 = 1$ or equivalently $t_0 t_1 = t_1 t_0$, and thus relation (1) holds in G.

We note that x, y and t_0 generate the whole group and so our group, let us call it H, is an image of G. Thus, $|G| \geq |H|$.

We also have that $|G| \leq 384 = |H|$.

Therefore, $|G| \leq 384 = |H| \leq |G|$,

which proves the isomorphism, that is $G \cong H$.

A Group G Over S_6

A symmetric presentation of the progenitor $2^{*6}: S_6$ is given by:

$<x, y, t \mid x^6, y^2, x\,y\,x^4\,y\,x^2\,y\,x^{-2}\,y\,x, (y\,x)^5, x^2\,y\,x^3\,y\,x^{-3}\,y\,x^{-3}\,y\,x,$
$t^2, (t, x\,y), (t, y^x) >$

where the action of the control group, N, on the symmetric generators is given by:

x = (0 1 2 3 4 5),

y = (1 2),

and $N^0 = <x\,y, x^{-1}y\,x>$

We factor the progenitor by the following relations

$(t_1\,t_2\,t_1\,(1\,4\,2))^2 = 1$ and $t_4\,t_2\,t_1\,t_4\,(1\,2\,4) = 1$

to get the homomorphic image G,

$$G \cong \frac{2^{*6} : S_6}{(t_1\,t_2\,t_1\,(1\,4\,2))^2 = 1, t_4\,t_2\,t_1\,t_4\,(1\,2\,4) = 1}$$

The index of N in G is 14.

Manual Double Coset Enumeration

Let $(1\,4\,2) = \pi$

Then, $(t_1\,t_2\,t_1\,(1\,4\,2))^2 = 1$

$\Rightarrow \quad t_1\,t_2\,t_1\,\pi\,t_1\,t_2\,t_1\,\pi = 1$

$\Rightarrow \quad \pi^2\,\pi^{-2}\,t_1\,t_2\,t_1\,\pi^2\,\pi^{-1}\,t_1\,t_2\,t_1\,\pi = 1$

$\Rightarrow \ \pi^2 (t_1 t_2 t_1)^{\pi^2} (t_1 t_2 t_1)^{\pi} = 1$

$\Rightarrow \ (1\,2\,4)\, t_2 t_4 t_2 t_4 t_1 t_4 = 1$

$\Rightarrow \ (1\,2\,4)\, t_2 t_4 t_2 = t_4 t_1 t_4$ (1)

$\Rightarrow \ N t_2 t_4 t_2 = N t_4 t_1 t_4$

Also, we can write

$N (t_2 t_4 t_2)^{(2\,3)} = N (t_4 t_1 t_4)^{(2\,3)}$

$\Rightarrow \ N t_3 t_4 t_3 = N t_4 t_1 t_4$

Similarly,

$N (t_2 t_4 t_2)^{(2\,4\,3)} = N (t_4 t_1 t_4)^{(2\,4\,3)}$

$\Rightarrow \ N t_4 t_3 t_4 = N t_3 t_1 t_3$

Similarly we will get equal cosets if we conjugate relation (1) by permutations in S_6.

Now the relation

$t_4 t_2 t_1 t_4 (1\,2\,4) = 1$

$\Rightarrow \ t_4 t_2 = (1\,4\,2)\, t_4 t_1$ (2)

$\Rightarrow \ N t_4 t_2 = N t_4 t_1$

Also, we can write

$N (t_4 t_2)^{(2\,3)} = N (t_4 t_1)^{(2\,3)}$

$\Rightarrow \ N t_4 t_3 = N t_4 t_1$

Similarly,

$N(t_4\ t_2)^{(2\ 5)} = N(t_4\ t_1)^{(2\ 5)}$

$\Rightarrow\ N t_4\ t_5 = N t_4\ t_1$

Thus we have,

$N t_4\ t_0 = N t_4\ t_1 = N t_4\ t_2 = N t_4\ t_3 = N t_4\ t_5$

Therefore the double coset [i j] contains 6 single cosets since every coset has 5 names.

Also, by (2), we can write

$N t_4\ t_2\ t_1 = N t_4$

Thus, [i j k] = [i] for all i, j, and k in {0,1,2,3,4,5}, since N is six transitive on T.

Now we consider the double coset [i j i],

By (2), we can write

$N t_4\ t_0\ t_4 = N t_4\ t_1\ t_4 = N t_4\ t_2\ t_4 = N t_4\ t_3\ t_4 = N t_4\ t_5\ t_4$

Also,

$N t_0\ t_1\ t_0 = N t_0\ t_2\ t_0 = N t_0\ t_3\ t_0 = N t_0\ t_4\ t_0 = N t_0\ t_5\ t_0$

$N t_1\ t_0\ t_1 = N t_1\ t_2\ t_1 = N t_1\ t_3\ t_1 = N t_1\ t_4\ t_1 = N t_1\ t_5\ t_1$

$N t_2\ t_0\ t_2 = N t_2\ t_1\ t_2 = N t_2\ t_3\ t_2 = N t_2\ t_4\ t_2 = N t_2\ t_5\ t_2$

$N t_3\ t_0\ t_3 = N t_3\ t_1\ t_3 = N t_3\ t_2\ t_3 = N t_3\ t_4\ t_3 = N t_3\ t_5\ t_3$

$N t_5\ t_0\ t_5 = N t_5\ t_1\ t_5 = N t_5\ t_2\ t_5 = N t_5\ t_3\ t_5 = N t_5\ t_4\ t_5$

And ,by (1), we got all of them to be equal.

Therefore the double coset [i j i] contains one single coset.

Thus, Cayley graph of G over N has the form:

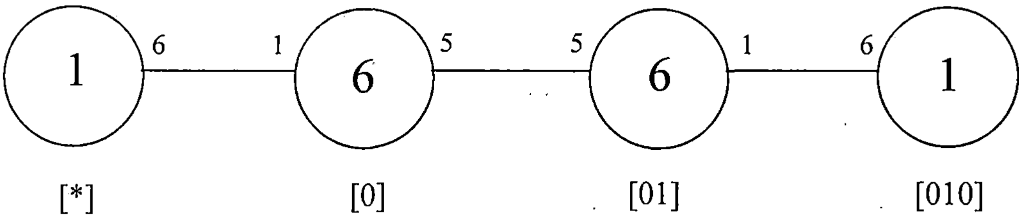

Figure 4. Cayley Graph of The Group G Over S_6

We obtain:

t_0: (* 0)(1 10)(2 20)(3 30)(4 40)(5 50)(01 010)

t_1: (* 1)(0 01)(2 21)(3 31)(4 41)(5 51)(10 101)

t_2: (* 2)(0 02)(1 12)(3 32)(4 42)(5 52)(21 212)

t_3: (* 3)(0 03)(1 13)(2 23)(4 43)(5 53)(31 313)

t_4: (* 4)(0 04)(1 14)(2 24)(3 34)(5 54)(41 414)

t_5: (* 5)(0 05)(1 15)(2 25)(3 35)(4 45)(51 515)

We rename the cosets as follows

*	0	1	2	3	4	5	01	12	23	34	45	50	010
1	2	3	4	5	6	7	8	9	10	11	12	13	14

Then, the six symmetric generators become:

t_0:(1 2)(3 9)(4 10)(5 11)(6 12)(7 13)(8 14)

t_1: (1 3)(2 8)(4 10)(5 11)(6 12)(7 13)(9 14)

t_2: (1 4)(2 8)(3 9)(5 11)(6 12)(7 13)(10 14)

t_3: (1 5)(2 8)(3 9)(4 10)(6 12)(7 13)(11 14)

t_4: (1 6)(2 8)(3 9)(4 10)(5 11)(7 13)(12 14)

t_5: (1 7)(2 8)(3 9)(4 10)(5 11)(6 12)(13 14)

Proof of Isomorphism

We obtained the collapsed Cayley graph from the information contained in the following computations

$$\frac{|N|}{|N|} + \frac{|N|}{|N^{(0)}|} + \frac{|N|}{|N^{(01)}|} + \frac{|N|}{|N^{(010)}|}$$

$$= \frac{720}{720} + \frac{720}{120} + \frac{720}{120} + \frac{720}{720}$$

$$= 1 + 6 + 6 + 1 = 14.$$

this tells us that the maximum possible index of N in G is 14. It follows that the order of the image group G is at most $|N| * 14 = 720 * 14 = 10{,}080$. The order of G can be confirmed by regarding G as a permutation group on the 14 cosets that we have found.

The action of the control group N on the cosets is

x: (2, 3, 4, 5, 6, 7) (8, 9, 10, 11, 13)

y: (3, 4) (9, 10)

We note that yx has order 5, and hence $N = \langle x, y \rangle \cong S_6$.

The action of the symmetric generator t_0 is given by:

$t_0 : (1, 2)(3, 9)(4, 10)(5, 11)(6, 12)(7, 13)(8, 14)$

The action of x and y, and hence N, on the symmetric generators is as follows:

$x : (t_0, t_1, t_2, t_3, t_4, t_5)$

$y : (t_1, t_2)$

Now we check our relations; that is

$t_0^{t_4 t_1 t_4 t_2 t_4 t_2} = (1, 2)(3, 9)(4, 10)(5, 11)(6, 12)(7, 13)(8, 14) = t_0$

$t_1^{t_4 t_1 t_4 t_2 t_4 t_2} = (1, 4)(2, 8)(3, 9)(5, 11)(6, 12)(7, 13)(10, 14) = t_2$

$t_2^{t_4 t_1 t_4 t_2 t_4 t_2} = (1, 6)(2, 8)(3, 9)(4, 10)(5, 11)(7, 13)(12, 14) = t_4$

$t_3^{t_4 t_1 t_4 t_2 t_4 t_2} = (1, 5)(2, 8)(3, 9)(4, 10)(6, 12)(7, 13)(11, 14) = t_3$

$t_4^{t_4 t_1 t_4 t_2 t_4 t_2} = (1, 3)(2, 8)(4, 10)(5, 11)(6, 12)(7, 13)(9, 14) = t_1$

$t_5^{t_4 t_1 t_4 t_2 t_4 t_2} = (1, 7)(2, 8)(3, 9)(4, 10)(5, 11)(6, 12)(13, 14) = t_5$

This means that $t_4\, t_1\, t_4\, t_2\, t_4\, t_2$ acts as the permutation (1 2 4) on the symmetric generators, that is $(1\,2\,4) = t_4\, t_1\, t_4\, t_2\, t_4\, t_2$, which proves relation (1).

Similarly,

$t_0^{t_4 t_2 t_1 t_4} = (1, 2)(3, 9)(4, 10)(5, 11)(6, 12)(7, 13)(8, 14) = t_0$

$t_1^{t_4 t_2 t_1 t_4} = (1, 6)(2, 8)(3, 9)(4, 10)(5, 11)(7, 13)(12, 14) = t_4$

$t_2^{t_4 t_2 t_1 t_4} = (1, 3)\,(2, 8)\,(4, 10)\,(5, 11)\,(6, 12)\,(7, 13)\,(9, 14) = t_1$

$t_3^{t_4 t_2 t_1 t_4} = (1, 5)\,(2, 8)\,(3, 9)\,(4, 10)\,(6, 12)\,(7, 13)\,(11, 14) = t_3$

$t_4^{t_4 t_2 t_1 t_4} = (1, 4)\,(2, 8)\,(3, 9)\,(5, 11)\,(6, 12)\,(7, 13)\,(10, 14) = t_2$

$t_5^{t_4 t_2 t_1 t_4} = (1, 7)\,(2, 8)\,(3, 9)\,(4, 10)\,(5, 11)\,(6, 12)\,(13, 14) = t_5$

Thus, $t_4\, t_2\, t_1\, t_4$ acts as the permutation (1 4 2) on the symmetric generators, that is $t_4\, t_2\, t_1\, t_4 = (1\,4\,2)$, which gives us relation (2).

The elements x, y and t_0 generates the whole group, and so our group, let us call it H, is an image of G. Thus, $|G| \geq |H|$.

We also have that $|G| \leq 10{,}080 = |H|$.

Therefore, $|G| \leq 10{,}080 = |H| \leq |G|$,

which proves the isomorphism, that is $G \cong H$.

CHAPTER FIVE

THE GROUP $PGL_2(11)$ OVER $L_2(5)$

A symmetric presentation of the group $2^{*6} : L_2(5)$ is given by: $< x, y, t \mid x^5, y^3, (x\,y)^2, t^2, (t,x), (t^{yx^2}, x\,y) >$

where the action of x and y on the symmetric generators is given by:

$x = (0\,1\,2\,3\,4)$,

$y = (\infty\,0\,1)\,(2\,4\,3)$,

and $N^0 = < x, y\,x^3\,y\,x^{-2}\,y^{-1} >$.

We factor the progenitor by the following relations $(x\,t_0)^4 = 1$ and $(\infty\,0)(1\,4) = (t_2\,t_3)^3$ to obtain the group G,

$$G \cong \frac{2^{*6} : L_2(5)}{(xt_0)^4, (\infty\,0)(1\,4) = (t_2\,t_3)^3}$$

Then, the index of N in G is 22.

Manual Double Coset Enumeration

The relation $(x\,t_0)^4 = 1$

$\Rightarrow \; x\,t_0\,x\,t_0\,x\,t_0\,x\,t_0 = 1$

$\Rightarrow \; x^4\,x^{-3}\,t_0\,x^3\,x^{-2}\,t_0\,x^2\,x^{-1}\,t_0\,x\,t_0 = 1$

$\Rightarrow \; (0\,4\,3\,2\,1)\,t_3\,t_2\,t_1\,t_0 = 1$

$\Rightarrow \; t_3\, t_2\, t_1\, t_0 = (0\ 1\ 2\ 3\ 4)$ (1)

$\Rightarrow \; t_3\, t_2 = (0\ 1\ 2\ 3\ 4)\, t_0\, t_1$

$\Rightarrow \; N\, t_3\, t_2 = N\, t_0\, t_1$

Note that,

$N\, (t_3\, t_2)^{(2\ 4)(3\ \infty)} = N\, (t_0\, t_1)^{(2\ 4)(3\ \infty)}$

$\Rightarrow \; N\, t_\infty\, t_4 = N\, t_0\, t_1$

Therefore, $N\, t_0\, t_1 = N\, t_3\, t_2 = N\, t_\infty\, t_4$

Since $L_2(5)$ is doubly transitive on $\{0,1,2,3,4,\infty\}$, the double coset [i j] contains 10 distinct single cosets because every coset has three names.

Furthermore, $N\, t_0\, t_1\, t_2 = N\, t_3\, t_2\, t_2$ (by 1)

$= N\, t_3$

Similarly,

$N\, t_0\, t_1\, t_4 = N\, t_\infty\, t_4\, t_4 = N\, t_\infty$.

$\Rightarrow$ [i j k] = [i] for all i, j and k in $\{0,1,2,3,4,\infty\}$.

The relation $(\infty\ 0)(1\ 4) = (t_2\, t_3)^3$

$\Rightarrow \; (\infty\ 0)(1\ 4) = t_2\, t_3\, t_2\, t_3\, t_2\, t_3$

$\Rightarrow \; (\infty\ 0)(1\ 4)\, t_3\, t_2\, t_3 = t_2\, t_3\, t_2$

$\Rightarrow \; N\, t_3\, t_2\, t_3 = N\, t_2\, t_3\, t_2$ (2)

$\Rightarrow$ $N t_i t_j t_i = N t_j t_i t_j$ for i and j in {0,1,2,3,4,∞} since $L_2(5)$ is doubly transitive on {0,1,2,3,4,∞}.

In particular, $N t_\infty t_0 t_\infty = N t_0 t_\infty t_0$.

Also, $N t_4 t_1 t_4 = N t_4 t_1 t_4 t_\infty t_2 t_2 t_\infty$

$= N t_4 (1\ 0\ 2\ \infty\ 4) t_2 t_\infty$

$= N t_1 t_2 t_\infty$

$= N t_\infty t_0 t_\infty$ (since $N t_1 t_2 = N t_\infty t_0$).

And $N t_\infty t_0 t_\infty = N t_\infty t_0 t_\infty t_3 t_1 t_1 t_3$

$= N t_\infty (1\ 3\ \infty\ 0\ 2) t_1 t_3$

$= N t_0 t_1 t_3$

$= N t_3 t_2 t_3$ (since $N t_0 t_1 = N t_3 t_2$).

Thus, we have

$N t_\infty t_0 t_\infty = N t_0 t_\infty t_0 = N t_3 t_2 t_3 = N t_2 t_3 t_2 = N t_4 t_1 t_4 = N t_1 t_4 t_1$

Therefore the double coset [i j i] contains 5 single cosets since each coset has 6 names.

Cayley graph of $PGL_2(11)$ over $L_2(5)$ is given below:

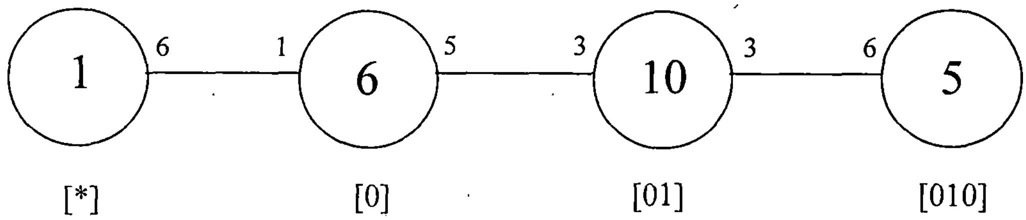

Figure 5. Cayley Graph of The Group $PGL_2(11)$ Over $L_2(5)$

For the action of the four symmetric generators on the cosets of $PGL_2(11)$ over $L_2(5)$, we label the 22 cosets as follows

1.	[]	12.	[4, 6]
2.	[5]	13.	[6, 1]
3.	[1]	14.	[5, 6]
4.	[6]	15.	[6, 4]
5.	[4]	16.	[1, 6]
6.	[2]	17.	[2, 6]
7.	[1, 5]	18.	[1, 6, 1]
8.	[6, 5]	19.	[2, 6, 2]
9.	[4, 5]	20.	[5, 6, 5]
10.	[3]	21.	[1, 5, 1]
11.	[3, 6]	22.	[4, 6, 4]

where

t_1: (1 3)(2 15)(4 13)(5 11)(6 9)(7 21)(8 22)(10 12)(14 19)(16 18)(17 20)

t_2: (1 6)(2 12)(3 8)(4 7)(5 14)(9 22)(10 15)(11 18)(13 20)(16 21)(17 19)

t_3: (1 10)(2 16)(3 14)(4 9)(5 8)(6 13)(7 18)(11 21)(12 19)(15 20)(17 22)

t_4: (1 5)(2 13)(3 17)(4 15)(6 16)(7 10)(8 18)(9 19)(11 20)(12 22)(14 21)

t_5: (1 2)(3 7)(4 8)(5 9)(6 11)(10 17)(12 18)(13 19)(14 20)(15 21)(16 22)

t_6: (1 4)(2 14)(3 16)(5 12)(6 17)(7 19)(8 20)(9 21)(10 11)(13 18)(15 22)

Proof of Isomorphism

We obtained the collapsed Cayley graph from the information contained in the following computations

$$\frac{|N|}{|N|} + \frac{|N|}{|N^{(0)}|} + \frac{|N|}{|N^{(01)}|} + \frac{|N|}{|N^{(010)}|}$$

$$= \frac{60}{60} + \frac{60}{10} + \frac{60}{6} + \frac{60}{12}$$

$$= 1 + 6 + 10 + 5 = 22$$

this tells us that the maximum possible index of N in G is 22. It follows that the order of the image group G is at most $|N| * 22 = 60 * 22 = 1320$. The order of G can be established by representing G as a permutation group on the 22 cosets that we have found.

We know that the action of the control group N on the cosets is well-defined and the actions of the symmetric generators on the cosets are also well-defined.

Now we will show that:

(i) t_∞ has just six images under conjugation by N.

(ii) The two additional relations hold within the symmetric group S_{22}.

However, we can directly verify that the image is the projective general linear group $PGL_2(11)$:

We first define the linear fractional permutations of the projective line $PGL_2(11)$ as follows

$$x: \left(\tau \mapsto \frac{\tau + 8}{9}\right) \equiv (2, 6, 4, 5, 10)(3, 11, 7, 9, 8)$$

$$y: \left(\tau \mapsto \frac{10\tau + 4}{\tau - 2}\right) \equiv (1, 8, 3)(2, 12, 10)(4, 11, 9)(5, 7, 6)$$

$$t_\infty: \left(\tau \mapsto -\tau + 2\right) \equiv (2, 11)(3, 10)(4, 9)(5, 8)(6, 7)$$

We note that xy has order 2, and hence $N = \langle x, y\rangle \cong L_2(5)$.

The permutation t_∞ has just six images under conjugation by N, namely:

$t_\infty = (2, 11)(3, 10)(4, 9)(5, 8)(6, 7)$

$t_0 = (1, 2)(3, 7)(4, 11)(5, 6)(9, 12)$

$t_1 = (1, 6)(4, 10)(5, 7)(8, 12)(9, 11)$

$t_2 = (1, 4)(2, 5)(3, 12)(7, 8)(9, 10)$

$t_3 = (1, 5)(2, 8)(3, 9)(6, 10)(11, 12)$

$t_4 = (1, 10)(2, 4)(3, 6)(7, 12)(8, 11)$

The action of x and y, and hence N, on these symmetric generators is as follows:

$x: (t_0, t_1, t_2, t_3, t_4)$

$y: (t_\infty, t_0, t_1)(t_2, t_4, t_3)$

Now we check our relations; that is

$$\begin{aligned} t_\infty^{t_3 t_2 t_1 t_0} &= (2, 11)(3, 10)(4, 9)(5, 8)(6, 7) \\ &= t_\infty \end{aligned}$$

$$\begin{aligned} t_0^{t_3 t_2 t_1 t_0} &= (1, 6)(4, 10)(5, 7)(8, 12)(9, 11) \\ &= t_1 \end{aligned}$$

$$\begin{aligned} t_1^{t_3 t_2 t_1 t_0} &= (1, 4)(2, 5)(3, 12)(7, 8)(9, 10) \\ &= t_2 \end{aligned}$$

$$\begin{aligned} t_2^{t_3 t_2 t_1 t_0} &= (1, 5)(2, 8)(3, 9)(6, 10)(11, 12) \\ &= t_3 \end{aligned}$$

$$\begin{aligned} t_3^{t_3 t_2 t_1 t_0} &= (1, 10)(2, 4)(3, 6)(7, 12)(8, 11) \\ &= t_4 \end{aligned}$$

$$t_4^{t_3 t_2 t_1 t_0} = (1, 2)(3, 7)(4, 11)(5, 6)(9, 12)$$
$$= t_0$$

Thus $t_3 t_2 t_1 t_0$ acts as the permutation (0 1 2 3 4) on the symmetric generators, that is $t_3 t_2 t_1 t_0 = (0\ 1\ 2\ 3\ 4)$, this proves that relation (1) holds in $PGL_2(11)$.

Similarly,

$$t_\infty^{(t_2 t_3)^3} = (1, 2)(3, 7)(4, 11)(5, 6)(9, 12)$$
$$= t_0$$

$$t_0^{(t_2 t_3)^3} = (2, 11)(3, 10)(4, 9)(5, 8)(6, 7)$$
$$= t_\infty$$

$$t_1^{(t_2 t_3)^3} = (1, 10)(2, 4)(3, 6)(7, 12)(8, 11)$$
$$= t_4$$

$$t_2^{(t_2 t_3)^3} = (1, 4)(2, 5)(3, 12)(7, 8)(9, 10)$$
$$= t_2$$

$$t_3^{(t_2 t_3)^3} = (1, 5)(2, 8)(3, 9)(6, 10)(11, 12)$$
$$= t_3$$

$$t_4^{(t_2 t_3)^3} = (1, 6)(4, 10)(5, 7)(8, 12)(9, 11)$$
$$= t_1$$

This means that $(t_2 t_3)^3$ acts as the permutation $(\infty\ 0)(1\ 4)$ on the symmetric generators, that is $(\infty\ 0)(1\ 4) = (t_2 t_3)^3$, hence our second relation holds in $PGL_2(11)$.

N is maximal in $PGL_2(11)$ and $t_\infty \notin N$, hence N and t_∞ generate the whole group, $PGL_2(11)$, and so $PGL_2(11)$ is an image of G. Thus $|G| \geq |PGL_2(11)|$.

Since we have that $|G| \leq 1320 = |PGL_2(11)|$

Therefore, $|G| \leq 1320 = |PGL_2(11)| \leq |G|$,

which proves the isomorphism, that is $G \cong PGL_2(11)$.

CHAPTER SIX

THE GROUP $PGL_2(7)$ OVER S_3

We consider

$$G \cong \frac{2^{*3} : S_3}{(xt_0)^8, (y^{x^{-1}} t_0)^7, (y^{x^{-1}} t_0t_2)^6, (y^{x^{-1}} t_0t_2t_0)^4}.$$

A presentation for G takes the form

$$<x, y, t \mid x^3, y^2, (xy)^2, (t,y), (xt_0)^8, (y^{x^{-1}} t_0)^7, (y^{x^{-1}} t_0t_2)^6, (y^{x^{-1}} t_0t_2t_0)^4>$$

where $x = (0\ 1\ 2)$,

$y = (1\ 2)$,

and $N^0 = <y>$.

The index of N in G is 56.

Manual Double Coset Enumeration

The relation

$$(x\,t_0)^8 = 1$$

$$\Rightarrow\ x\,t_0\ x\,t_0\ x\,t_0\ x\,t_0\ x\,t_0\ x\,t_0\ x\,t_0\ x\,t_0 = 1$$

$$\Rightarrow\ x^2\ x^{-1}\ t_0\ x\,t_0\ x^3\ x^{-2}\ t_0\ x^2\ x^{-1}\ t_0\ x\,t_0\ x^3\ x^{-2}\ t_0\ x^2\ x^{-1}\ t_0\ x\,t_0 = 1$$

$$\Rightarrow\ x^2\ t_0^{\,x}\ t_0\ t_0^{\,x^2}\ t_0^{\,x}\ t_0\ t_0^{\,x^2}\ t_0^{\,x}\ t_0 = 1$$

$$\Rightarrow\ (0\ 2\ 1)\ t_1\ t_0\ t_2\ t_1\ t_0\ t_2\ t_1\ t_0 = 1$$

$\Rightarrow\ (0\,2\,1)\, t_1\, t_0\, t_2\, t_1 = t_0\, t_1\, t_2\, t_0$ (1)

$\Rightarrow\ N t_1\, t_0\, t_2\, t_1 = N t_0\, t_1\, t_2\, t_0$

Since S_3 is three transitive on {0,1,2}, then the double coset [i j k i] contains 3 distinct single cosets because every coset has two names.

Let $y^{x^{-1}} = \pi$. Thus, the relation

$(y^{x^{-1}}\, t_0)^7 = 1$ becomes $(\pi\, t_0)^7 = 1$

$(y^{x^{-1}}\, t_0)^7 = 1 \Rightarrow \pi t_0\, \pi t_0\, \pi t_0\, \pi t_0\, \pi t_0\, \pi t_0\, \pi t_0 = 1$

$\Rightarrow\ \pi\, t_0\, t_0^{\pi}\, t_0\, t_0^{\pi}\, t_0\, t_0^{\pi}\, t_0 = 1$ (since $\pi = (0\,1)$, so $\pi = \pi^{-1}$)

$\Rightarrow\ (0\,1)\, t_0\, t_1\, t_0\, t_1\, t_0\, t_1\, t_0 = 1$

$\Rightarrow\ (0\,1)\, t_0\, t_1\, t_0\, t_1 = t_0\, t_1\, t_0$ (2)

$\Rightarrow\ N t_i\, t_j\, t_i\, t_j = N t_i\, t_j\, t_i$ for i and j in {0,1,2} since S_3 is three transitive on {0,1,2}.

Now the relation

$(\pi\, t_0\, t_2)^6 = 1$

$\pi t_0\, t_2\, \pi t_0\, t_2\, \pi t_0\, t_2\, \pi t_0\, t_2\, \pi t_0\, t_2\, \pi t_0\, t_2 = 1$

$\Rightarrow\ \pi t_0\, \pi \pi t_2\, \pi t_0\, t_2\, \pi t_0\, \pi \pi t_2\, \pi t_0\, t_2\, \pi t_0\, \pi \pi t_2\, \pi t_0\, t_2 = 1$

$\Rightarrow\ t_0^{\pi}\, t_2^{\pi}\, t_0\, t_2\, t_0^{\pi}\, t_2^{\pi}\, t_0\, t_2\, t_0^{\pi}\, t_2^{\pi}\, t_0\, t_2 = 1$ (since $\pi = (0\,1)$, so $\pi = \pi^{-1}$)

$\Rightarrow\ t_1\, t_2\, t_0\, t_2\, t_1\, t_2\, t_0\, t_2\, t_1\, t_2\, t_0\, t_2 = 1$

$\Rightarrow\ t_1\, t_2\, t_0\, t_2\, t_1\, t_2 = t_2\, t_0\, t_2\, t_1\, t_2\, t_0$ (3)

Finally, the relation

$(\pi\, t_0\, t_2\, t_0)^4 = 1$

$\pi\, t_0\, t_2\, t_0\, \pi\, t_0\, t_2\, t_0\, \pi\, t_0\, t_2\, t_0\, \pi\, t_0\, t_2\, t_0 = 1$

$\Rightarrow\ \pi\, t_0\, \pi\, \pi\, t_2\, \pi\, \pi\, t_0\, \pi\, t_0\, t_2\, t_0\, \pi\, t_0\, \pi\, \pi\, t_2\, \pi\, \pi\, t_0\, \pi\, t_0\, t_2\, t_0 = 1$

$\Rightarrow\ t_0^{\pi}\, t_2^{\pi}\, t_0^{\pi}\, t_0\, t_2\, t_0\, t_0^{\pi}\, t_2^{\pi}\, t_0^{\pi}\, t_0\, t_2\, t_0 = 1$ (since $\pi = (0\ 1)$, so $\pi = \pi^{-1}$)

$\Rightarrow\ t_1\, t_2\, t_1\, t_0\, t_2\, t_0\, t_1\, t_2\, t_1\, t_0\, t_2\, t_0 = 1$

$\Rightarrow\ t_1\, t_2\, t_1\, t_0\, t_2\, t_0 = t_0\, t_2\, t_0\, t_1\, t_2\, t_1$ (4)

By (1), we can write

$(0\ 2\ 1)\, t_1\, t_0\, t_2\, t_1\, t_0 = t_0\, t_1\, t_2$

$\Rightarrow\ N t_1\, t_0\, t_2\, t_1\, t_0 = N t_0\, t_1\, t_2$

$\Rightarrow$ [i j k i j] = [j i k] for i and j in {0,1,2} since

S_3 is three transitive on {0,1,2}.

Also, by (1), we can write

$(0\ 2\ 1)\, t_1\, t_0\, t_2\, t_1\, t_2 = t_0\, t_1\, t_2\, t_0\, t_2$

$\Rightarrow\ N t_1\, t_0\, t_2\, t_1\, t_2 = N t_0\, t_1\, t_2\, t_0\, t_2$

Thus, the double coset [i j k i k] contains 3 distinct single cosets since each coset has two names.

Now, note that

$$t_0\ t_1\ t_0\ t_2\ t_1 = t_0\ (0\ 1\ 2)\ (0\ 2\ 1)\ t_1\ t_0\ t_2\ t_1$$

$$= t_0\ (0\ 1\ 2)\ t_0\ t_1\ t_2\ t_0 \qquad \text{(by 1)}$$

$$= (0\ 1\ 2)\ t_1\ t_0\ t_1\ t_2\ t_0$$

$$\Rightarrow\ N t_0\ t_1\ t_0\ t_2\ t_1 = N t_1\ t_0\ t_1\ t_2\ t_0 \qquad (*)$$

Since S_3 is three transitive on {0,1,2}, then the double coset [i j i k j] contains 3 distinct single cosets because every coset has two names.

Also, note that

$$t_0\ t_1\ t_2\ t_1\ t_2 = t_0\ t_1\ t_2\ t_1\ t_2\ t_1\ t_2\ t_2\ t_1$$

$$= t_0\ (1\ 2)\ t_1\ t_2\ t_1 \qquad \text{(by 2)}$$

$$= (1\ 2)\ t_0\ t_1\ t_2\ t_1$$

$$\Rightarrow\ N t_0\ t_1\ t_2\ t_1\ t_2 = N t_0\ t_1\ t_2\ t_1$$

$\Rightarrow$ [i j k j k] = [i j k j] for i and j in {0,1,2} since S_3 is three transitive on {0,1,2}.

Now,

$$t_0\ t_1\ t_2\ t_0\ t_2\ t_0 = t_0\ t_1\ (0\ 2)\ t_2\ t_0\ t_2 \qquad \text{(by 2)}$$

$$= (0\ 2)\ t_2\ t_1\ t_2\ t_0\ t_2$$

$\Rightarrow$ [i j k i k i] = [i j i k i] for i and j in {0,1,2} since

S_3 is three transitive on {0,1,2}.

Also,

$$t_0\, t_1\, t_2\, t_0\, t_2\, t_1 = (0\ 2\ 1)\, t_1\, t_0\, t_2\, t_1\, t_2\, t_1 \qquad \text{(by 1)}$$

$$= (0\ 2\ 1)\ t_1\, t_0\, (1\ 2)\, t_2\, t_1\, t_2 \qquad \text{(by 2)}$$

$$= (0\ 2\ 1)\,(1\ 2)\ t_2\, t_0\, t_2\, t_1\, t_2$$

$$= (0\ 1)\ t_2\, t_0\, t_2\, t_1\, t_2$$

$\Rightarrow$ [i j k i k j] = [i j i k i] for i and j in {0,1,2} since S_3 is three transitive on {0,1,2}.

Now, note that

$$t_0\, t_1\, t_2\, t_1\, t_0\, t_1 = t_0\, t_1\, t_2\, (0\ 1)\, t_1\, t_0\, t_1\, t_0 \qquad \text{(by 2)}$$

$$= (0\ 1)\, t_1\, t_0\, t_2\, t_1\, t_0\, t_1\, t_0$$

$$= (0\ 1)\,(0\ 1\ 2)\,(0\ 2\ 1)\, t_1\, t_0\, t_2\, t_1\, t_0\, t_1\, t_0$$

$$= (0\ 1)\,(0\ 1\ 2)\, t_0\, t_1\, t_2\, t_0\, t_0\, t_1\, t_0 \qquad \text{(by 1)}$$

$$= (0\ 2)\, t_0\, t_1\, t_2\, t_1\, t_0$$

$$\Rightarrow\ N\, t_0\, t_1\, t_2\, t_1\, t_0\, t_1 = N\ t_0\, t_1\, t_2\, t_1\, t_0$$

$\Rightarrow$ [i j k j i j] = [i j k j i] for i and j in {0,1,2} since S_3 is three transitive on {0,1,2}.

Also, note that:

$$t_1\, t_2\, t_0\, t_2\, t_1\, t_0 = t_1\, t_2\, (1\ 2)\,(1\ 2)\, t_0\, t_2\, t_1\, t_0$$

$= t_1\ t_2\ (1\ 2)(0\ 1)\ t_2\ t_0\ t_1\ t_2$ (by 1)

$= t_1\ t_2\ (0\ 1\ 2)\ t_2\ t_0\ t_1\ t_2$

$= (0\ 1\ 2)\ t_2\ t_0\ t_2\ t_0\ t_1\ t_2$

$= (0\ 1\ 2)(0\ 2)\ t_2\ t_0\ t_2\ t_1\ t_2$ (by 2)

$= (0\ 1)\ t_2\ t_0\ t_2\ t_1\ t_2$

$\Rightarrow\ N t_1\ t_2\ t_0\ t_2\ t_1\ t_0 = N\ t_2\ t_0\ t_2\ t_1\ t_2$

$\Rightarrow$ [i j k j i k] = [i j i k i] for i and j in {0,1,2} since S_3 is three transitive on {0,1,2}.

Now, by (3), we have

$t_2\ t_0\ t_2\ t_1\ t_2\ t_0 = t_1\ t_2\ t_0\ t_2\ t_1\ t_2$

$\Rightarrow\ N t_2\ t_0\ t_2\ t_1\ t_2\ t_0 = N t_1\ t_2\ t_0\ t_2\ t_1\ t_2$

$\Rightarrow$ [i j i k i j] = [i j k j i j]

But we have showed that [i j k j i j] = [i j k j i]

Thus, [i j i k i j] = [i j k j i] for i and j in {0,1,2} since S_3 is three transitive on {0,1,2}.

Now,

$t_0\ t_1\ t_0\ t_2\ t_0\ t_2 = t_0\ t_1\ (0\ 2)\ t_0\ t_2\ t_0$ (by 2)

$= (0\ 2)\ t_2\ t_1\ t_0\ t_2\ t_0$

$\Rightarrow$ [i j i k i k] = [i j k i k] for i and j in {0,1,2} since S_3 is three transitive on {0,1,2}.

Also,

$$t_0\ t_1\ t_0\ t_2\ t_1\ t_0 = t_0\ t_1\ (0\ 1\ 2)\ t_2\ t_0\ t_1\ t_2 \qquad \text{(by 1)}$$

$$= (0\ 1\ 2)\ t_1\ t_2\ t_2\ t_0\ t_1\ t_2$$

$$= (0\ 1\ 2)\ t_1\ t_0\ t_1\ t_2$$

$\Rightarrow$ [i j i k j i] = [i j i k] for i and j in {0,1,2} since S_3 is three transitive on {0,1,2}.

Note that, by (*), we can write

$$N t_0\ t_1\ t_0\ t_2\ t_1\ t_2 = N t_1\ t_0\ t_1\ t_2\ t_0\ t_2$$

$$N t_2\ t_0\ t_2\ t_1\ t_0\ t_1 = N t_0\ t_2\ t_0\ t_1\ t_2\ t_1$$

$$N t_1\ t_2\ t_1\ t_0\ t_2\ t_0 = N t_2\ t_1\ t_2\ t_0\ t_1\ t_0$$

Also, by (4), we have

$$t_1\ t_2\ t_1\ t_0\ t_2\ t_0 = t_0\ t_2\ t_0\ t_1\ t_2\ t_1$$

$$t_0\ t_1\ t_0\ t_2\ t_1\ t_2 = t_2\ t_1\ t_2\ t_0\ t_1\ t_0$$

$$t_2\ t_0\ t_2\ t_1\ t_0\ t_1 = t_1\ t_0\ t_1\ t_2\ t_0\ t_2$$

Thus, the double coset [i j i k j k] consists of a single coset.

Cayley Graph of $PGL_2(7)$ over S_3 will take the form

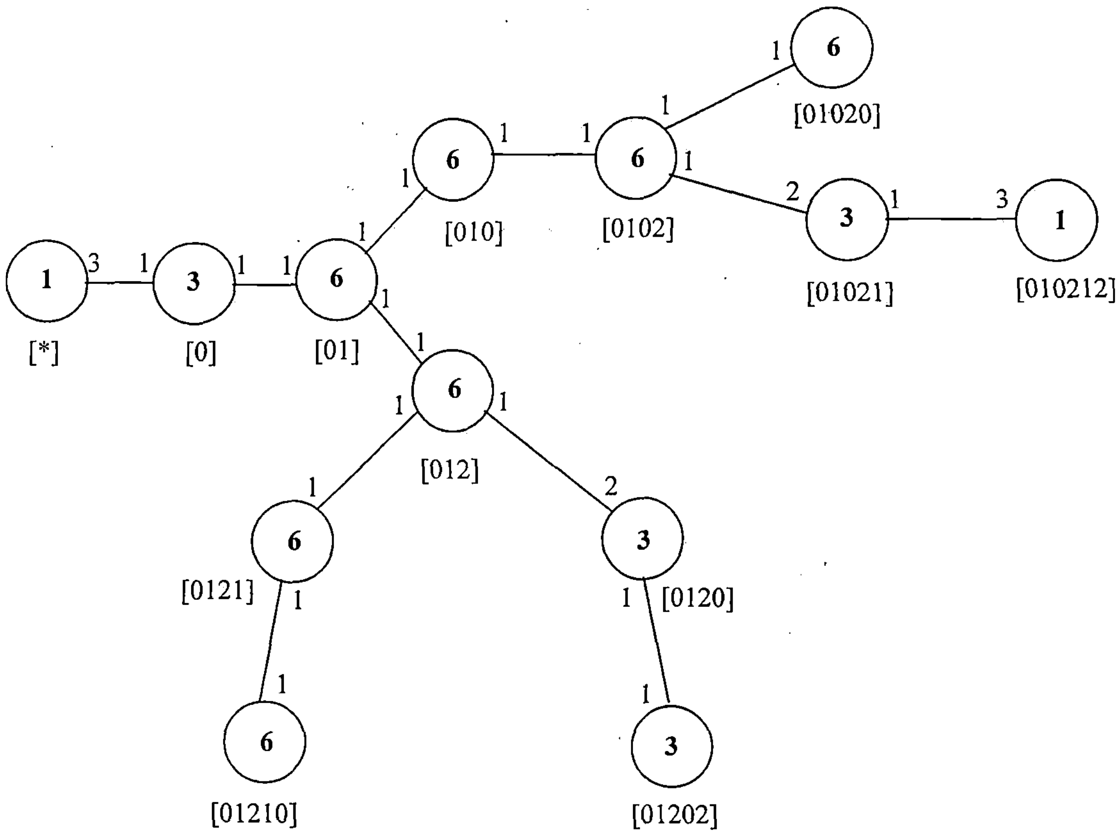

Figure 6. Cayley Graph of The Group $PGL_2(7)$ Over S_3

The action of the symmetric generators is given below:

t_0: (* 0)(1 10)(2 20)(21 210)(02 020)(01 010)(12 120)
(021 0210)(102 1020)(101 1010)(202 2020)(212 2120)
(121 1210)(201 2010)(012 0120)(1201 12010)(2101 21010)
(0212 02120)(0201 02010)(1012 10120)(2021 20210)
(0102 01020) (0121 01210)(20121 201210)(01202 012020)
(1202 12020)(10201 102010)(21012 210120)
(12102 121020)(20212 202120)(20102 201020)

t_1: (* 1)(0 01)(2 21)(10 101)(20 201)(02 021)(12 121)
(010 0101)(210 2101)(020 0201)(120 1201)(102 1021)
(212 2121)(202 2021)(012 0121)(2012 20121)(1020 10201)
(2120 21201)(1210 12101)(2010 20101)(0212 02121)
(1012 10121)(0102 01021)(1202 12021)(12010 120101)
(02120 021201)(20210 202101)(01020 010201)
(01210 012101)(01202 012021)(21012 210121)

t_2: (* 2)(0 02)(1 12)(10 102)(20 202)(21 212)(01 012)
(210 2102)(020 0202)(010 0102)(120 1202)(021 0210)
(101 1012)(121 1212)(201 2012)(1020 10202)(2120 21202)
(1210 12102)(2010 20102)(0120 01202)(2101 21012)
(0201 02012)(2021 20212)(0121 01212)(12010 120102)
(02120 021202)(02010 020102)(01021 010212)
(01210 012102)(20121 201212)(12021 120212)

Then we rename the cosets as follows:

1	[]	7	[2, 1]
2	[0]	8	[0, 2]
3	[1]	9	[0, 1]
4	[2]	10	[1, 2]
5	[1, 0]	11	[2, 1, 0]
6	[2, 0]	12	[0, 2, 0]

13 [0, 1, 0]
14 [1, 2, 0]
15 [0, 2, 1]
16 [1, 0, 2]
17 [1, 0, 1]
18 [2, 1, 2]
19 [1, 2, 1]
20 [2, 0, 2]
21 [2, 0, 1]
22 [0, 1, 2]
23 [2, 0, 1, 2]
24 [1, 0, 2, 0]
25 [2, 1, 2, 0]
26 [1, 2, 1, 0]
27 [2, 0, 1, 0]
28 [0, 1, 2, 0]
29 [1, 2, 0, 1]
30 [2, 1, 0, 1]
31 [0, 2, 1, 2]
32 [0, 2, 0, 1]
33 [1, 0, 1, 2]
34 [2, 0, 2, 1]
35 [0, 1, 0, 2]
36 [0, 1, 2, 1]
37 [1, 2, 0, 2]
38 [1, 2, 0, 1, 0]
39 [0, 2, 1, 2, 0]
40 [0, 2, 0, 1, 0]
41 [0, 1, 0, 2, 1]
42 [2, 0, 2, 1, 0]
43 [0, 1, 0, 2, 0]
44 [0, 1, 2, 1, 0]
45 [2, 0, 1, 2, 1]
46 [0, 1, 2, 0, 2]
47 [1, 0, 2, 0, 1]
48 [2, 1, 0, 1, 2]
49 [1, 0, 1, 2, 1]
50 [2, 1, 2, 0, 2]
51 [1, 2, 1, 0, 2]
52 [1, 2, 1, 0, 1]
53 [2, 0, 2, 1, 2]
54 [1, 2, 0, 2, 1]
55 [2, 0, 1, 0, 2]
56 [0, 1, 0, 2, 1, 2]

Thus, the action of the symmetric generators becomes:

t_0: (1, 2)(3, 5)(4, 6)(7, 11)(8, 12)(9, 13)(10, 14)
(15, 23)(16, 24)(18, 25)(19, 26)(21, 27)(22, 28)
(29, 38)(31, 39)(32, 40)(33, 41)(34, 42)(35, 43)
(36, 44)(45, 52)(46, 50)(48, 49)(51, 56)(53, 54)

t_1: (1, 3)(2, 9)(4, 7)(5, 17)(6, 21)(8, 15)(10, 19)
(11, 30)(12, 32)(14, 29)(16, 28)(20, 34)(22, 36)
(23, 45)(24, 47)(25, 51)(26, 52)(33, 49)(35, 41)
(37, 54)(38, 40)(39, 50)(42, 56)(43, 55)(46, 53)

t_2: (1, 4)(2, 8)(3, 10)(5, 16)(6, 20)(7, 18)(9, 22)
(11, 29)(13, 35)(14, 37)(15, 31)(17, 33)(21, 23)
(25, 50)(26, 51)(27, 55)(28, 46)(30, 48)(32, 42)
(34, 53)(38, 43)(40, 47)(41, 56)(44, 52)(45, 49)

Proof of Isomorphism

We obtained the collapsed Cayley graph from the information contained in the following computations

$$\frac{|N|}{|N|}+\frac{|N|}{|N^{(0)}|}+\frac{|N|}{|N^{(01)}|}+\frac{|N|}{|N^{(010)}|}+\frac{|N|}{|N^{(012)}|}+\frac{|N|}{|N^{(0102)}|}+\frac{|N|}{|N^{(0120)}|}+\frac{|N|}{|N^{(0121)}|}$$
$$+\frac{|N|}{|N^{(01020)}|}+\frac{|N|}{|N^{(01021)}|}+\frac{|N|}{|N^{(01202)}|}+\frac{|N|}{|N^{(01210)}|}+\frac{|N|}{|N^{(010212)}|}$$

$$= \frac{6}{6} + \frac{6}{2} + \frac{6}{1} + \frac{6}{1} + \frac{6}{1} + \frac{6}{1} + \frac{6}{2} + \frac{6}{1} + \frac{6}{1} + \frac{6}{2} + \frac{6}{2} + \frac{6}{1} + \frac{6}{6}$$

$$= 1 + 3 + 6 + 6 + 6 + 6 + 3 + 6 + 6 + 3 + 3 + 6 + 1 = 56 .$$

This tells us that the maximum possible index of N in G is 56. It follows that the order of the image group G is at most $|N| * 56 = 6 * 56 = 336$. The order of G can be established by considering G as a permutation group on the 56 cosets that we have found.

The action of the control group N on the cosets is

x: (2, 3, 4) (5, 7, 8) (6, 9, 10) (13, 19, 20) (12, 17, 18) (11, 15, 16)
(14, 21, 22) (23, 28, 29) (24, 30, 31) (27, 36, 37) (25, 32, 33)
(26, 34, 35) (38, 45, 46) (39, 47, 48) (44, 54, 55) (40, 49, 50)
(43, 52, 53) (41, 51, 42)

y: (3, 4) (5, 6) (7, 10) (8, 9) (11, 14) (12, 13) (15, 22) (16, 21) (17, 20)
(18, 19) (23, 28) (24, 27) (25, 26) (30, 37) (31, 36) (32, 35) (33, 34)
(39, 44) (40, 43) (41, 42) (45, 46) (47, 55) (48, 54) (49, 53) (50, 52)

The action of the symmetric generator t_0 on the cosets is:

t_0: (1, 2) (3, 5) (4, 6) (7, 11) (8, 12) (9, 13) (10, 14) (15, 23)(16, 24)
(18, 25)(19, 26)(21, 27)(22, 28) (29, 38) (31, 39) (32, 40) (33, 41)
(34, 42) (35, 43) (36, 44)(45, 52)(46, 50)(48, 49)(51, 56)(53, 54)

The action of x and y, and hence N, on these symmetric generators is as follows:

$x: (t_0, t_1, t_2)$

$y: (t_1, t_2)$

We note that xy has order 2, and hence $N = \langle x, y \rangle \cong S_3$.

Now we check our relations; that is

$$t_0^{t_1 t_0 t_2 t_1 t_0 t_2 t_1 t_0} = (1,3)(2,9)(4,7)(5,17)(6,21)(8,15)(10,19)(11,30)(12,32)(14,29)(16,28)(20,34)(22,36)(23,45)(24,47)(25,51)(26,52)(33,49)(35,41)(37,54)(38,40)(39,50)(42,56)(43,55)(46,53) = t_1$$

$$t_1^{t_1 t_0 t_2 t_1 t_0 t_2 t_1 t_0} = (1,4)(2,8)(3,10)(5,16)(6,20)(7,18)(9,22)(11,29)(13,35)(14,37)(15,31)(17,33)(21,23)(25,50)(26,51)(27,55)(28,46)(30,48)(32,42)(34,53)(38,43)(40,47)(41,56)(44,52)(45,49) = t_2$$

$$t_2^{t_1 t_0 t_2 t_1 t_0 t_2 t_1 t_0} = (1,2)(3,5)(4,6)(7,11)(8,12)(9,13)(10,14)(15,23)(16,24)(18,25)(19,26)(21,27)(22,28)(29,38)(31,39)(32,40)(33,41)(34,42)(35,43)(36,44)(45,52)(46,50)(48,49)(51,56)(53,54) = t_0$$

This means that $t_1 t_0 t_2 t_1 t_0 t_2 t_1 t_0$ acts as the permutation (0 1 2) on the symmetric generators, that is

$t_1 t_0 t_2 t_1 t_0 t_2 t_1 t_0 = (0\ 1\ 2)$, which proves our first relation.

Similarly,

$$t_0^{t_0 t_1 t_0 t_1 t_0 t_1 t_0} = (1,3)(2,9)(4,7)(5,17)(6,21)(8,15)(10,19)(11,30)(12,32)(14,29)(16,28)(20,34)(22,36)(23,45)(24,47)(25,51)(26,52)(33,49)(35,41)(37,54)(38,40)(39,50)(42,56)(43,55)(46,53) = t_1$$

$$t_1^{t_0 t_1 t_0 t_1 t_0 t_1 t_0} = (1,2)(3,5)(4,6)(7,11)(8,12)(9,13)(10,14)(15,23)(16,24)(18,25)(19,26)(21,27)(22,28)(29,38)(31,39)(32,40)(33,41)(34,42)(35,43)(36,44)(45,52)(46,50)(48,49)(51,56)(53,54) = t_0$$

$$t_2^{t_0 t_1 t_0 t_1 t_0 t_1 t_0} = (1,4)(2,8)(3,10)(5,16)(6,20)(7,18)(9,22)(11,29)(13,35)(14,37)(15,31)(17,33)(21,23)(25,50)(26,51)(27,55)(28,46)(30,48)(32,42)(34,53)(38,43)(40,47)(41,56)(44,52)(45,49) = t_2$$

This means that $t_0\, t_1\, t_0\, t_1\, t_0\, t_1\, t_0$ acts as the permutation (0 1) on the symmetric generators, that is $t_0\, t_1\, t_0\, t_1\, t_0\, t_1\, t_0 = (0\ 1)$, which gives us our second relation.

Also,

$$t_0^{t_1 t_2 t_0 t_2 t_1 t_2 t_0 t_2 t_1 t_2 t_0 t_2} = (1,2)(3,5)(4,6)(7,11)(8,12)(9,13)(10,14)(15,23)(16,24)(18,25)(19,26)(21,27)(22,28)(29,38)(31,39)(32,40)(33,41)(34,42)(35,43)(36,44)(45,52)(46,50)(48,49)(51,56)(53,54) = t_0$$

$$t_1^{t_1 t_2 t_0 t_2 t_1 t_2 t_0 t_2 t_1 t_2 t_0 t_2} = (1,3)(2,9)(4,7)(5,17)(6,21)(8,15)(10,19)(11,30)(12,32)(14,29)(16,28)(20,34)(22,36)(23,45)(24,47)(25,51)(26,52)(33,49)(35,41)(37,54)(38,40)(39,50)(42,56)(43,55)(46,53) = t_1$$

$$t_2^{t_1 t_2 t_0 t_2 t_1 t_2 t_0 t_2 t_1 t_2 t_0 t_2} = (1,4)(2,8)(3,10)(5,16)(6,20)(7,18)(9,22)(11,29)(13,35)(14,37)(15,31)(17,33)(21,23)(25,50)(26,51)(27,55)(28,46)(30,48)(32,42)(34,53)(38,43)(40,47)(41,56)(44,52)(45,49) = t_2$$

This means that $t_1\, t_2\, t_0\, t_2\, t_1\, t_2\, t_0\, t_2\, t_1\, t_2\, t_0\, t_2$ fixes all the symmetric generators, so it acts as the identity on t_0, t_1, t_2, that is $t_1\, t_2\, t_0\, t_2\, t_1\, t_2\, t_0\, t_2\, t_1\, t_2\, t_0\, t_2 = 1$ or equivalently $t_1\, t_2\, t_0\, t_2\, t_1\, t_2 = t_2\, t_0\, t_2\, t_1\, t_2\, t_0$, and that proves relation (3).

Further,

$$t_0^{t_1 t_2 t_1 t_0 t_2 t_0 t_1 t_2 t_1 t_0 t_2 t_0} = (1,2)(3,5)(4,6)(7,11)(8,12)(9,13)(10,14)(15,23)(16,24)(18,25)(19,26)(21,27)(22,28)(29,38)(31,39)(32,40)(33,41)(34,42)(35,43)(36,44)(45,52)(46,50)(48,49)(51,56)(53,54) = t_0$$

$$t_1^{t_1 t_2 t_1 t_0 t_2 t_0 t_1 t_2 t_1 t_0 t_2 t_0} = (1,3)(2,9)(4,7)(5,17)(6,21)(8,15)(10,19)(11,30)(12,32)(14,29)(16,28)(20,34)(22,36)(23,45)(24,47)(25,51)(26,52)(33,49)(35,41)(37,54)(38,40)(39,50)(42,56)(43,55)(46,53) = t_1$$

$$t_2^{t_1 t_2 t_1 t_0 t_2 t_0 t_1 t_2 t_1 t_0 t_2 t_0} = (1,4)(2,8)(3,10)(5,16)(6,20)(7,18)(9,22)(11,29)(13,35)(14,37)(15,31)(17,33)(21,23)(25,50)(26,51)(27,55)(28,46)(30,48)(32,42)(34,53)(38,43)(40,47)(41,56)(44,52)(45,49) = t_2$$

This means that $t_1 t_2 t_1 t_0 t_2 t_0 t_1 t_2 t_1 t_0 t_2 t_0$ fixes all the symmetric generators, so it acts as identity on t_0, t_1, t_2, that is $t_1 t_2 t_1 t_0 t_2 t_0 t_1 t_2 t_1 t_0 t_2 t_0 = 1$ or equivalently $t_1 t_2 t_1 t_0 t_2 t_0 = t_0 t_2 t_0 t_1 t_2 t_1$, and that proves relation (4).

The elements x, y and t_0 generates the whole group, $PGL_2(7)$, and so $PGL_2(7)$ is an image of G. Thus, $|G| \geq |PGL_2(7)|$. We also have that $|G| \leq 336 = |PGL_2(7)|$. Therefore, $|G| \leq 336 = |PGL_2(7)| \leq |G|$, which proves the isomorphism, that is $G \cong PGL_2(7)$.

CHAPTER SEVEN

THE GROUP $L_2(11)$ OVER A_4

A symmetric presentation of the group $2^{*4} : A_4$ is given by: $< x, y, t \mid x^3, y^3, (x\,y)^2, t^2, (t,x) >$

where the action of x and y on the symmetric generators is given by:

$x = (1\,2\,3)$,

$y = (0\,1\,2)$,

and $N^0 = < x >$.

We factor the progenitor by the following relations

$(y\,t_0)^{11} = 1$, $(x\,y\,t_0)^5 = 1$ and $(y\,t_0\,t_3)^5 = 1$

To obtain the group G,

$$G \cong \frac{2^{*4} : A_4}{(y\,t_0)^{11}, (x\,y\,t_0)^5, (y\,t_0\,t_3)^5}$$

Then, the index of N in G is 55.

Manual Double Coset Enumeration

The relation $(y\,t_0)^{11} = 1$

$(y\,t_0)^{11} = 1$

$\Rightarrow\; y\,t_0\,y\,t_0\,y\,t_0\,y\,t_0\,y\,t_0\,y\,t_0\,y\,t_0\,y\,t_0\,y\,t_0\,y\,t_0\,y\,t_0 = 1$

$\Rightarrow y^2\, y^{-1}t_0\, yt_0\, y^3\, y^{-2}t_0\, y^2\, y^{-1}t_0\, yt_0\, y^3\, y^{-2}t_0\, y^2\, y^{-1}t_0\, y\, t_0\, y^3\, y^{-2}\, t_0\, y^2\, y^{-1}t_0\, yt_0=1$

$\Rightarrow \;\; y^2\, t_0^{\,y}\, t_0\, t_0^{\,y^2}\, t_0^{\,y}\, t_0\, t_0^{\,y^2}\, t_0^{\,y}\, t_0\, t_0^{\,y^2}\, t_0^{\,y}\, t_0 = 1$

$\Rightarrow \;\; (0\,2\,1)\, t_1\, t_0\, t_2\, t_1\, t_0\, t_2\, t_1\, t_0\, t_2\, t_1\, t_0 = 1$

$\Rightarrow \;\; (0\,2\,1)\, t_1\, t_0\, t_2\, t_1\, t_0 = t_0\, t_1\, t_2\, t_0\, t_1\, t_2$ (1)

$\Rightarrow \;\; N\, t_1\, t_0\, t_2\, t_1\, t_0 = N\, t_0\, t_1\, t_2\, t_0\, t_1\, t_2$

Now consider the relation:

$(x\, y\, t_0)^5 = 1$

Let $xy = \pi$. Thus, the relation

$(x\, y\, t_0)^5 = 1$ becomes $(\pi\, t_0)^5 = 1$

$(\pi\, t_0)^5 = 1 \;\Rightarrow\; \pi\, t_0\, \pi\, t_0\, \pi\, t_0\, \pi\, t_0\, \pi\, t_0 = 1$

$\Rightarrow \;\; \pi\, t_0\, t_0^{\pi}\, t_0\, t_0^{\pi}\, t_0 = 1$ (since $\pi = (0\,1)(2\,3)$, so $\pi = \pi^{-1}$)

$\Rightarrow \;\; (0\,1)(2\,3)\, t_0\, t_1\, t_0\, t_1\, t_0 = 1$

$\Rightarrow \;\; (0\,1)(2\,3) = t_0\, t_1\, t_0\, t_1\, t_0$ (2)

$\Rightarrow \;\; N\, t_0\, t_1\, t_0 = N\, t_0\, t_1$

Thus, [i j i]=[i j] for i and j in {0,1,2,3} since A_4 is doubly transitive on {0,1,2,3}.

The relation

$(y\, t_0\, t_3)^5 = 1$

$\Rightarrow \;\; y\, t_0\, t_3\, y\, t_0\, t_3\, y\, t_0\, t_3\, y\, t_0\, t_3\, y\, t_0\, t_3 = 1$

$\Rightarrow\ y^2\, y^{-1}\, t_0\, t_3\, y\, t_0\, t_3\, y^3\, y^{-2}\, t_0\, t_3\, y^2\, y^{-1}\, t_0\, t_3\, y\, t_0\, t_3 = 1$

$\Rightarrow\ y^2\, t_0^{\,y}\, t_3^{\,y}\, t_0\, t_3\, t_0^{\,y^2}\, t_3^{\,y^2}\, t_0^{\,y}\, t_3^{\,y}\, t_0\, t_3 = 1$

$\Rightarrow\ (0\ 2\ 1)\, t_1\, t_3\, t_0\, t_3\, t_2\, t_3\, t_1\, t_3\, t_0\, t_3 = 1$

$\Rightarrow\ (0\ 2\ 1) = t_3\, t_0\, t_3\, t_1\, t_3\, t_2\, t_3\, t_0\, t_3\, t_1$ (3)

Note that, the double coset [i j k] contains 12 distinct single cosets and also that the double coset [i j l] contains 12 distinct single cosets.

Now,

$t_0\, t_1\, t_2\, t_0 = (0\ 1)\,(2\ 3)\, t_0\, t_1\, t_0\, t_2\, t_0$ (by 2)

$= (0\ 1)\,(2\ 3)\, t_0\, t_1\, (0\ 2)\,(1\ 3)\, t_0\, t_2$ (by 2)

$= (0\ 1)\,(2\ 3)\,(0\ 2)\,(1\ 3)\, t_2\, t_3\, t_0\, t_2$

$= (0\ 3)\,(1\ 2)\, t_2\, t_3\, t_0\, t_2$

$\Rightarrow\ N\, t_0\, t_1\, t_2\, t_0 = N\, t_2\, t_3\, t_0\, t_2$

Therefore, the double coset [i j k i] contains 6 distinct single cosets since each coset has two names.

Using (3), we can write

$(0\ 2\ 1)\, t_1\, t_3\, t_0\, t_3\, t_2 = t_3\, t_0\, t_3\, t_1\, t_3$

$\Rightarrow\ (0\ 2\ 1)\, t_1\, (0\ 3)\,(1\ 2)\, t_3\, t_0\, t_2 = (0\ 3)\,(1\ 2)\, t_3\, t_0\, t_1\, t_3$ (by 2)

$\Rightarrow\ (0\ 2\ 1)\,(0\ 3)\,(1\ 2)\, t_2\, t_3\, t_0\, t_2 = (0\ 3)\,(1\ 2)\, t_3\, t_0\, t_1\, t_3$

$\Rightarrow\ (0\ 1\ 3)\, t_2\, t_3\, t_0\, t_2 = (0\ 3)\,(1\ 2)\, t_3\, t_0\, t_1\, t_3$

$$\Rightarrow \; N t_2 t_3 t_0 t_2 = N t_3 t_0 t_1 t_3 \qquad (*)$$

Therefore, [i j l i] = [i j k i] for i, j, k and l in {0,1,2,3}.

Now,

$$\begin{aligned}
t_0 t_1 t_2 t_3 &= t_0 t_1 t_2 t_3 t_2 t_0 t_0 t_2 \\
&= t_0 (0\,3\,1)(0\,1\,3) t_1 t_2 t_3 t_2 t_0 t_0 t_2 \\
&= t_0 (0\,3\,1) t_2 t_3 t_2 t_1 t_2 t_0 t_2 && \text{(by 3)} \\
&= (0\,3\,1) t_3 t_2 t_3 t_2 t_1 t_2 t_0 t_2 \\
&= (0\,3\,1)(0\,1)(2\,3) t_3 t_1 (0\,2)(1\,3) t_2 t_0 && \text{(by 2)} \\
&= (0\,3\,1)(0\,1)(2\,3)(0\,2)(1\,3) t_1 t_3 t_2 t_0 \\
&= (1\,3\,2) t_1 t_3 t_2 t_0
\end{aligned}$$

Also,

$$\begin{aligned}
t_1 t_3 t_2 t_0 &= t_1 t_3 t_2 t_0 t_2 t_1 t_1 t_2 \\
&= t_1 (0\,3\,1)(0\,1\,3) t_3 t_2 t_0 t_2 t_1 t_1 t_2 \\
&= t_1 (0\,3\,1) t_2 t_0 t_2 t_3 t_2 t_1 t_2 && \text{(by 3)} \\
&= (0\,3\,1) t_0 t_2 t_0 t_2 t_3 t_2 t_1 t_2 \\
&= (0\,3\,1)(0\,2)(1\,3) t_0 t_3 (0\,3)(1\,2) t_2 t_1 && \text{(by 2)} \\
&= (0\,3\,1)(0\,2)(1\,3)(0\,3)(1\,2) t_3 t_0 t_2 t_1 \\
&= (0\,2\,3) t_3 t_0 t_2 t_1
\end{aligned}$$

$\Rightarrow \; N t_0 \; t_1 \; t_2 \; t_3 = N t_1 \; t_3 \; t_2 \; t_0 = N t_3 \; t_0 \; t_2 \; t_1$ (**)

Thus, the double coset [i j k l] contains 4 distinct single cosets because every coset has three names.

Using (3), we can write

$(0\,2\,1)\, t_1 \; t_3 \; t_0 = t_3 \; t_0 \; t_3 \; t_1 \; t_3 \; t_2 \; t_3$

$\Rightarrow \; (0\,2\,1)\, t_1 \; t_3 \; t_0 \; t_2 = t_3 \; t_0 \; t_3 \; t_1 \; t_3 \; t_2 \; t_3 \; t_2$

$\Rightarrow \; (0\,2\,1)\, t_1 \; t_3 \; t_0 \; t_2 = (0\,3)\,(1\,2)\, t_3 \; t_0 \; t_1 \; (0\,1)\,(2\,3)\, t_3$

$\Rightarrow \; (0\,2\,1)\, t_1 \; t_3 \; t_0 \; t_2 = (0\,3)\,(1\,2)\,(0\,1)\,(2\,3)\, t_2 \; t_1 \; t_0 \; t_3$

$\Rightarrow \; (0\,2\,1)\, t_1 \; t_3 \; t_0 \; t_2 = (0\,2)\,(1\,3)\, t_2 \; t_1 \; t_0 \; t_3$

Also, by (3), we can write

$(0\,3\,2)\, t_2 \; t_1 \; t_0 = t_1 \; t_0 \; t_1 \; t_2 \; t_1 \; t_3 \; t_1$

$\Rightarrow \; (0\,3\,2)\, t_2 \; t_1 \; t_0 \; t_3 = t_1 \; t_0 \; t_1 \; t_2 \; t_1 \; t_3 \; t_1 \; t_3$

$\Rightarrow \; (0\,3\,2)\, t_2 \; t_1 \; t_0 \; t_3 = (0\,1)\,(2\,3)\, t_1 \; t_0 \; t_2 \; (0\,2)\,(1\,3)\, t_1$

$\Rightarrow \; (0\,3\,2)\, t_2 \; t_1 \; t_0 \; t_3 = (0\,1)\,(2\,3)\,(0\,2)\,(1\,3)\, t_3 \; t_2 \; t_0 \; t_1$

$\Rightarrow \; (0\,3\,2)\, t_2 \; t_1 \; t_0 \; t_3 = (0\,3)\,(1\,2)\, t_3 \; t_2 \; t_0 \; t_1$

$\Rightarrow \; N t_1 \; t_3 \; t_0 \; t_2 = N t_2 \; t_1 \; t_0 \; t_3 = N t_3 \; t_2 \; t_0 \; t_1$ (***)

Thus, the double coset [i j l k] contains 4 distinct single cosets since each coset has three names.

By (*),

$N t_2 \; t_3 \; t_0 \; t_2 \; t_3 = N t_3 \; t_0 \; t_1$

Therefore, [i j k i j] = [i j l] for i, j, k and l in {0,1,2,3}.

Also, by (*) we can write

$$N\, t_2\, t_3\, t_0\, t_2 = N\, t_1\, t_2\, t_3\, t_1$$

$$\Rightarrow\; N\, t_2\, t_3\, t_0\, t_2\, t_1 = N\, t_1\, t_2\, t_3$$

Therefore, [i j k i l] = [i j l] for i, j, k and l in {0,1,2,3}.

By (**),

$$N\, t_0\, t_1\, t_2\, t_3\, t_0 = N\, t_1\, t_3\, t_2$$

Therefore, [i j k l i] = [i j k] for i, j, k and l in {0,1,2,3}.

Also, by (**)

$$N\, t_1\, t_3\, t_2\, t_0\, t_3 = N\, t_0\, t_1\, t_2$$

Therefore, [i j k l j] = [i j k] for i, j, k and l in {0,1,2,3}.

By (***), we can write

$$N\, t_2\, t_1\, t_0\, t_3\, t_2 = N\, t_1\, t_3\, t_0$$

Therefore, [i j l k i] = [i j l] for i, j, k and l in {0,1,2,3}.

Also, by (***)

$$N\, t_1\, t_3\, t_0\, t_2\, t_3 = N\, t_2\, t_1\, t_0$$

Therefore, [i j l k j] = [i j l] for i, j, k and l in {0,1,2,3}.

Cayley graph of $L_2(11)$ over A_4 is given below:

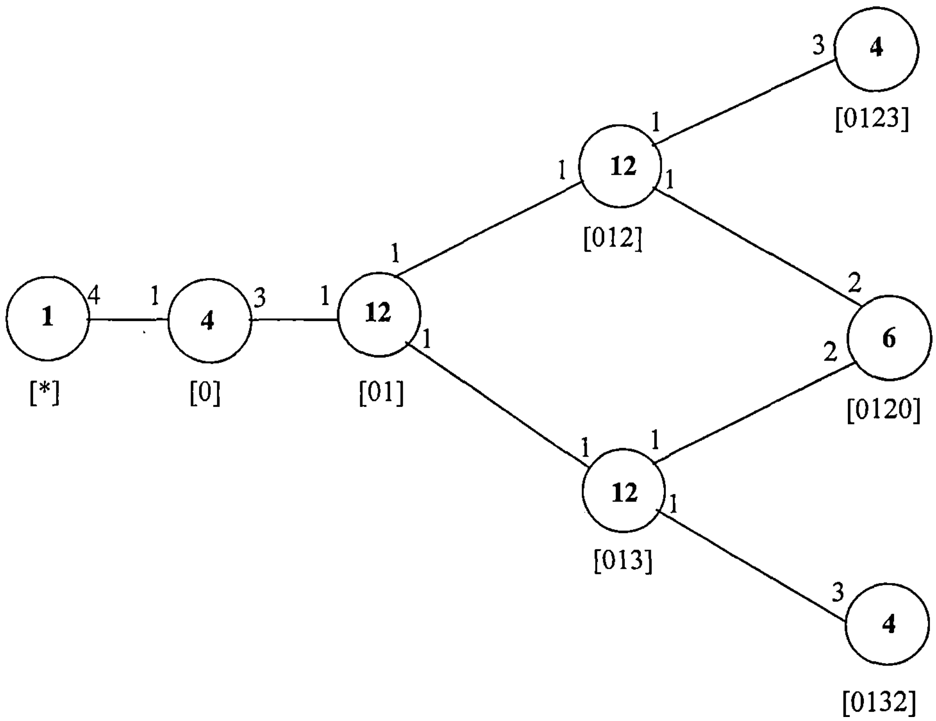

Figure 7. Cayley Graph of The Group $L_2(11)$ Over A_4

For the action of the four symmetric generators on the cosets of $L_2(11)$ over A_4, we label the 55 cosets as follows:

1 []	4 [2]
2 [4]	5 [1, 4]
3 [1]	6 [3]

7 [2, 4]

8 [2, 1]

9 [3, 4]

10 [4, 2]

11 [4, 1]

12 [1, 2]

13 [3, 2]

14 [2, 1, 4]

15 [1, 3]

16 [3, 1]

17 [4, 3]

18 [2, 3]

19 [1, 2, 4]

20 [3, 2, 4]

21 [4, 2, 1]

22 [1, 3, 4]

23 [1, 4, 2]

24 [3, 1, 4]

25 [2, 3, 4]

26 [2, 4, 1]

27 [4, 1, 2]

28 [3, 4, 1]

29 [3, 1, 2]

30 [4, 3, 2]

31 [2, 1, 3, 2]

32 [4, 1, 3]

33 [2, 3, 1]

34 [2, 4, 3]

35 [3, 4, 2]

36 [3, 2, 1]

37 [4, 3, 1]

38 [1, 3, 2]

39 [1, 4, 3]

40 [4, 2, 3]

41 [4, 1, 2, 4]

42 [1, 2, 3]

43 [4, 3, 2, 1]

44 [1, 4, 3, 1]

45 [4, 2, 3, 4]

46 [1, 3, 2, 1]

47 [4, 2, 1, 3]

48 [4, 3, 1, 2]

49 [2, 1, 3]

50 [4, 3, 1, 4]

51 [4, 1, 2, 3]

52 [4, 1, 3, 2]

53 [1, 3, 4, 2]

54 [4, 2, 3, 1]

55 [1, 2, 4, 3]

where

$t_0 =$ (1, 2)(3, 5)(4, 7)(6, 9)(8, 14)(12, 19)(13, 20)(15, 22)(16, 24)(18, 25)(21, 31) (23, 35)(26, 28)(27, 41)(29, 43)(30, 44)(32, 46)(33, 47)(34, 39)(36, 48)(37, 50)(38, 51)(40, 45)(42, 52)(49, 54)(53, 55)

$t_1 =$ (1, 3)(2, 11)(4, 8)(6, 16)(7, 26)(9, 28)(10, 21)(13, 36)(14, 24)(17, 37)(18, 33)(19, 50)(20, 53)(22, 31)(23, 45)(25, 55)(27, 29)(30, 43)(32, 49)(34, 52)(35, 51)(38, 46)(39, 44)(40, 54)(41, 42)(47, 48)

$t_2 =$ (1, 4)(2, 10)(3, 12)(5, 23)(6, 13)(9, 35)(11, 27)(14, 44)(15, 38)(16, 29)(17, 30)(19, 20)(21, 36)(22, 53)(24, 55)(25, 41)(26, 46)(28, 47)(31, 49)(32, 52)(33, 45)(34, 50)(37, 48)(39, 54)(40, 42)(43, 51)

$t_3 =$ (1, 6)(2, 17)(3, 15)(4, 18)(5, 39)(7, 34)(8, 49)(10, 40)(11, 32)(12, 42)(14, 53)(19, 55)(20, 46)(21, 47)(22, 25)(23, 43)(24, 45)(26, 48)(27, 51)(28, 41)(29, 50)(30, 38)(31, 35)(33, 37)(36, 44)(52, 54)

Proof of Isomorphism

We obtained the collapsed Cayley graph from the information contained in the following computations

$$\frac{|N|}{|N|}+\frac{|N|}{|N^{(0)}|}+\frac{|N|}{|N^{(01)}|}+\frac{|N|}{|N^{(012)}|}+\frac{|N|}{|N^{(013)}|}+\frac{|N|}{|N^{(0123)}|}+\frac{|N|}{|N^{(0120)}|}+\frac{|N|}{|N^{(0132)}|}$$

$$=\frac{12}{12}+\frac{12}{3}+\frac{12}{1}+\frac{12}{1}+\frac{12}{1}+\frac{12}{3}+\frac{12}{2}+\frac{12}{3}$$

$$=1+4+12+12+12+4+6+4=55$$

this tells us that the maximum possible index of N in G is 55. It follows that the order of the image group G is at most $|N| * 55 = 12 * 55 = 660$. The order of G can be established by regarding G as a permutation group on the 55 cosets that we have found.

The action of the control group N on the cosets is

x: (3, 4, 6) (5, 7, 9) (8, 13, 15) (10, 17, 11) (12, 18, 16) (14, 20, 22) (19, 25, 24) (21, 30, 32) (23, 34, 28) (26, 35, 39) (27, 40, 37) (29, 42, 33) (36, 38, 49) (31, 44, 46) (41, 45, 50) (43, 52, 47) (48, 51, 54)

y: (2, 3, 4) (5, 8, 10) (7, 11, 12) (9, 16, 13) (14, 21, 23) (15, 18, 17) (19, 26, 27) (20, 28, 29) (22, 33, 30) (24, 36, 35) (25, 37, 38) (31, 45, 44) (32, 42, 34) (39, 49, 40) (41, 50, 46) (43, 53, 47) (48, 51, 55)

The action of x and y, and hence N, on the symmetric generators is as follows:

$x: (t_1, t_2, t_3)$

$y: (t_0, t_1, t_2)$

The action of the symmetric generator t_0 on the cosets is:

$t_0:$ (1, 2) (3, 5) (4, 7) (6, 9) (8, 14) (12, 19) (13, 20) (15, 22) (16, 24) (18, 25) (21, 31) (23, 35) (26, 28) (27, 41) (29, 43) (30, 44) (32, 46) (33, 47) (34, 39) (36, 48) (37, 50) (38, 51) (40, 45) (42, 52) (49, 54) (53, 55)

We note that xy has order 2, and hence $N = \langle x, y\rangle \cong A_4$.

Now we check our relations; that is

$t_0^{t_1 t_0 t_2 t_1 t_0 t_2 t_1 t_0 t_2 t_1 t_0}$ = (1, 3) (2, 11) (4, 8) (6, 16) (7, 26) (9, 28) (10, 21) (13, 36) (14, 24) (17, 37) (18, 33) (19, 50) (20, 53) (22, 31) (23, 45) (25, 55) (27, 29) (30, 43) (32, 49) (34, 52) (35, 51) (38, 46) (39, 44) (40, 54) (41, 42) (47, 48)

$= t_1$

$t_1^{t_1 t_0 t_2 t_1 t_0 t_2 t_1 t_0 t_2 t_1 t_0}$ = (1, 4) (2, 10) (3, 12) (5, 23) (6, 13) (9, 35) (11, 27) (14, 44) (15, 38) (16, 29) (17, 30) (19, 20) (21, 36) (22, 53) (24, 55) (25, 41) (26, 46) (28, 47) (31, 49) (32, 52) (33, 45) (34, 50) (37, 48) (39, 54) (40, 42) (43, 51)

$= t_2$

$t_2^{t_1 t_0 t_2 t_1 t_0 t_2 t_1 t_0 t_2 t_1 t_0}$ = (1, 2) (3, 5) (4, 7) (6, 9) (8, 14) (12, 19) (13, 20) (15, 22) (16, 24) (18, 25) (21, 31) (23, 35) (26, 28) (27, 41) (29, 43) (30, 44) (32, 46) (33, 47) (34, 39) (36, 48) (37, 50) (38, 51) (40, 45) (42, 52) (49, 54) (53, 55)

$= t_0$

$$t_3^{t_1 t_0 t_2 t_1 t_0 t_2 t_1 t_0 t_2 t_1 t_0} = (1, 6)(2, 17)(3, 15)(4, 18)(5, 39)(7, 34)(8, 49)(10, 40)(11, 32)(12, 42)(14, 53)(19, 55)(20, 46)(21, 47)(22, 25)(23, 43)(24, 45)(26, 48)(27, 51)(28, 41)(29, 50)(30, 38)(31, 35)(33, 37)(36, 44)(52, 54) = t_3$$

This means that $t_1\ t_0\ t_2\ t_1\ t_0\ t_2\ t_1\ t_0\ t_2\ t_1\ t_0$ acts as the permutation (0 1 2) on the symmetric generators, that is

$t_1\ t_0\ t_2\ t_1\ t_0\ t_2\ t_1\ t_0\ t_2\ t_1\ t_0 = (0\ 1\ 2)$, which proves relation (1).

Similarly,

$$t_0^{t_0 t_1 t_0 t_1 t_0} = (1, 3)(2, 11)(4, 8)(6, 16)(7, 26)(9, 28)(10, 21)(13, 36)(14, 24)(17, 37)(18, 33)(19, 50)(20, 53)(22, 31)(23, 45)(25, 55)(27, 29)(30, 43)(32, 49)(34, 52)(35, 51)(38, 46)(39, 44)(40, 54)(41, 42)(47, 48) = t_1$$

$$t_1^{t_0 t_1 t_0 t_1 t_0} = (1, 2)(3, 5)(4, 7)(6, 9)(8, 14)(12, 19)(13, 20)(15, 22)(16, 24)(18, 25)(21, 31)(23, 35)(26, 28)(27, 41)(29, 43)(30, 44)(32, 46)(33, 47)(34, 39)(36, 48)(37, 50)(38, 51)(40, 45)(42, 52)(49, 54)(53, 55) = t_0$$

$$t_2^{t_0 t_1 t_0 t_1 t_0} = (1, 6)(2, 17)(3, 15)(4, 18)(5, 39)(7, 34)(8, 49)(10, 40)(11, 32)(12, 42)(14, 53)(19, 55)(20, 46)(21, 47)(22, 25)(23, 43)(24, 45)(26, 48)(27, 51)(28, 41)(29, 50)(30, 38)(31, 35)(33, 37)(36, 44)(52, 54) = t_3$$

$$t_3^{t_0 t_1 t_0 t_1 t_0} = (1, 4)(2, 10)(3, 12)(5, 23)(6, 13)(9, 35)(11, 27)(14, 44)(15, 38)(16, 29)(17, 30)(19, 20)(21, 36)(22, 53)(24, 55)(25, 41)(26, 46)(28, 47)(31, 49)(32, 52)(33, 45)(34, 50)(37, 48)(39, 54)(40, 42)(43, 51) = t_2$$

This means that $t_0\ t_1\ t_0\ t_1\ t_0$ acts as the permutation

(0 1)(2 3) on the symmetric generators, that is

$t_0\, t_1\, t_0\, t_1\, t_0 = (0\ 1)(2\ 3)$, which tells us that relation (2) holds in $L_2(11)$.

Also,

$$t_0^{t_3 t_0 t_3 t_1 t_3 t_2 t_3 t_0 t_3 t_1} = (1,4)(2,10)(3,12)(5,23)(6,13)(9,35)(11,27)(14,44)(15,38)(16,29)(17,30)(19,20)(21,36)(22,53)(24,55)(25,41)(26,46)(28,47)(31,49)(32,52)(33,45)(34,50)(37,48)(39,54)(40,42)(43,51) = t_2$$

$$t_1^{t_3 t_0 t_3 t_1 t_3 t_2 t_3 t_0 t_3 t_1} = (1,2)(3,5)(4,7)(6,9)(8,14)(12,19)(13,20)(15,22)(16,24)(18,25)(21,31)(23,35)(26,28)(27,41)(29,43)(30,44)(32,46)(33,47)(34,39)(36,48)(37,50)(38,51)(40,45)(42,52)(49,54)(53,55) = t_0$$

$$t_2^{t_3 t_0 t_3 t_1 t_3 t_2 t_3 t_0 t_3 t_1} = (1,3)(2,11)(4,8)(6,16)(7,26)(9,28)(10,21)(13,36)(14,24)(17,37)(18,33)(19,50)(20,53)(22,31)(23,45)(25,55)(27,29)(30,43)(32,49)(34,52)(35,51)(38,46)(39,44)(40,54)(41,42)(47,48) = t_1$$

$$t_3^{t_3 t_0 t_3 t_1 t_3 t_2 t_3 t_0 t_3 t_1} = (1,6)(2,17)(3,15)(4,18)(5,39)(7,34)(8,49)(10,40)(11,32)(12,42)(14,53)(19,55)(20,46)(21,47)(22,25)(23,43)(24,45)(26,48)(27,51)(28,41)(29,50)(30,38)(31,35)(33,37)(36,44)(52,54) = t_3$$

This means that $t_3\, t_0\, t_3\, t_1\, t_3\, t_2\, t_3\, t_0\, t_3\, t_1$ acts as the permutation (0 2 1) on the symmetric generators, that is

$t_3\, t_0\, t_3\, t_1\, t_3\, t_2\, t_3\, t_0\, t_3\, t_1 = (0\ 2\ 1)$, and that proves relation (3).

The elements x, y and t_0 generate the whole group, $L_2(11)$, and so $L_2(11)$ is an image of G. Thus, $|G| \geq |L_2(11)|$.

We also have that $|G| \leq 660 = |L_2(11)|$.

Therefore, $|G| \leq 660 = |L_2(11)| \leq |G|$,

which proves the isomorphism, that is $G \cong L_2(11)$.

CHAPTER EIGHT

THE GROUP $L_2(23)$ OVER S_4

A presentation of the group $2^{*4}:S_4$ is given by

$<x, y, t \mid x^4, y^2, (x\,y)^3, t^2, (t, y), (t\wedge x, y)>$

where the action of x and y on the symmetric generators is given by:

$x = (0\,1\,2\,3)$,

$y = (2\,3)$,

and $N^0 = <y, x\,y\,x^{-1}>$.

We factor the progenitor by the following relations

$(xt_0)^{11} = 1$, $(y\,x\,t_0)^{11} = 1$ and $y = (t_0t_1)^3$, to obtain the group G,

$$G \cong \frac{2^{*4}:S_4}{(xt_0)^{11}, (y\,x\,t_0)^{11}, y = (t_0t_1)^3}$$

The index of N in G is 253.

Manual Double Coset Enumeration

The relation

$(x\,t_0)^{11} = 1$

$\Rightarrow\; x\,t_0\,x\,t_0\,x\,t_0\,x\,t_0\,x\,t_0\,x\,t_0\,x\,t_0\,x\,t_0\,x\,t_0\,x\,t_0\,x\,t_0 = 1$

$$\Rightarrow\ x^3x^{-2}\,t_0x^2\,x^{-1}\,t_0x\,t_0\,x^4x^{-3}\,t_0x^3x^{-2}t_0x^2x^{-1}\,t_0x\,t_0\,x^4x^{-3}t_0x^3x^{-2}t_0x^2x^{-1}\,t_0x\,t_0=1$$

$$\Rightarrow\ x^3\,t_0^{x^2}\,t_0^{x}\,t_0\,t_0^{x^3}\,t_0^{x^2}\,t_0^{x}\,t_0\,t_0^{x^3}\,t_0^{x^2}\,t_0^{x}\,t_0=1$$

$$\Rightarrow\ x^3\,t_2\,t_1\,t_0\,t_3\,t_2\,t_1\,t_0\,t_3\,t_2\,t_1\,t_0=1$$

$$\Rightarrow\ x^3\,t_2\,t_1\,t_0\,t_3\,t_2\,t_1=t_0\,t_1\,t_2\,t_3\,t_0 \qquad (1)$$

$$\Rightarrow\ N\,t_2\,t_1\,t_0\,t_3\,t_2\,t_1=N\,t_0\,t_1\,t_2\,t_3\,t_0$$

The relation

$(y\,x\,t_0)^{11}=1$ with $y\,x=\pi$ becomes

$(\pi\,t_0)^{11}=1$

$$\Rightarrow\ \pi\,t_0\,\pi\,t_0\,\pi\,t_0\,\pi\,t_0\,\pi\,t_0\,\pi\,t_0\,\pi\,t_0\,\pi\,t_0\,\pi\,t_0\,\pi\,t_0\,\pi\,t_0=1$$

$$\Rightarrow\ \pi^2\,\pi^{-1}t_0\pi\,t_0\,\pi^3\,\pi^{-2}t_0\pi^2\,\pi^{-1}t_0\pi\,t_0\,\pi^3\,\pi^{-2}t_0\pi^2\,\pi^{-1}t_0\pi\,t_0\,\pi^3\,\pi^{-2}t_0\pi^2\,\pi^{-1}t_0\pi\,t_0=1$$

$$\Rightarrow\ \pi^2\,t_0^{\pi}\,t_0\,t_0^{\pi^2}\,t_0^{\pi}\,t_0\,t_0^{\pi^2}\,t_0^{\pi}\,t_0\,t_0^{\pi^2}\,t_0^{\pi}\,t_0=1$$

$$\Rightarrow\ \pi^2\,t_1\,t_0\,t_2\,t_1\,t_0\,t_2\,t_1\,t_0\,t_2\,t_1\,t_0=1$$

$$\Rightarrow\ \pi^2\,t_1\,t_0\,t_2\,t_1\,t_0\,t_2=t_0\,t_1\,t_2\,t_0\,t_1 \qquad (2)$$

$$\Rightarrow\ N\,t_1\,t_0\,t_2\,t_1\,t_0\,t_2=N\,t_0\,t_1\,t_2\,t_0\,t_1$$

Now, the relation

$y=(t_0\,t_1)^3$

$$\Rightarrow\ (2\ 3)=t_0\,t_1\,t_0\,t_1\,t_0\,t_1$$

$$\Rightarrow\ (2\ 3)\,t_0\,t_1\,t_0=t_1\,t_0\,t_1 \qquad (3)$$

$\Rightarrow \; N t_0 t_1 t_0 = N t_1 t_0 t_1$

$\Rightarrow \; N t_i t_j t_i = N t_j t_i t_j$ for i and j in {0,1,2,3} since S_4 is four transitive on {0,1,2,3}.

Now, note that

$$t_0 t_1 t_2 t_1 = t_0 (0\,3) t_2 t_1 t_2 \qquad \text{(by 3)}$$

$$= (0\,3) t_3 t_2 t_1 t_2$$

$\Rightarrow \; N t_0 t_1 t_2 t_1 = N t_3 t_2 t_1 t_2$

Since S_4 is four transitive on {0, 1, 2, 3}, then the double coset [i j k j] contains 12 distinct single cosets since each coset has two names.

Also, note that

$$t_0 t_1 t_0 t_2 t_0 = t_0 t_1 (1\,3) t_2 t_0 t_2 \qquad \text{(by 3)}$$

$$= (1\,3) t_0 t_3 t_2 t_0 t_2$$

$\Rightarrow \; N t_0 t_1 t_0 t_2 t_0 = N t_0 t_3 t_2 t_0 t_2$

$\Rightarrow$ [i j i k i] = [i j k i k] for i, j and k in {0, 1, 2, 3} since S_4 is four transitive on {0, 1, 2, 3}.

Also,

$$t_0 t_1 t_0 t_2 t_0 = (2\,3) t_1 t_0 t_1 t_2 t_0 \qquad \text{(by 3)}$$

$\Rightarrow\ N t_0 t_1 t_0 t_2 t_0 = N t_1 t_0 t_1 t_2 t_0$

$\Rightarrow$ [i j i k i] = [i j i k j] for i, j and k in {0, 1, 2, 3} since S_4 is four transitive on {0, 1, 2, 3}.

And,

$$t_0 t_1 t_0 t_2 t_1 = (2\ 3) t_1 t_0 t_1 t_2 t_1 \qquad \text{(by 3)}$$

$$= (2\ 3) t_1 t_0 (0\ 3) t_2 t_1 t_2 \qquad \text{(by 3)}$$

$$= (0\ 3\ 2) t_1 t_3 t_2 t_1 t_2$$

$\Rightarrow\ N t_0 t_1 t_0 t_2 t_1 = N t_1 t_3 t_2 t_1 t_2$

$\Rightarrow$ [i j i k j] = [i j k i k] for i, j and k in {0, 1, 2, 3} since S_4 is four transitive on {0, 1, 2, 3}.

Note that

$$t_0 t_1 t_2 t_1 t_0 = t_0 (0\ 3) t_2 t_1 t_2 t_0 \qquad \text{(by 3)}$$

$$= (0\ 3) t_3 t_2 t_1 t_2 t_0$$

$\Rightarrow\ N t_0 t_1 t_2 t_1 t_0 = N t_3 t_2 t_1 t_2 t_0$

$\Rightarrow$ [i j k j i] = [i j k j l] for i, j and k in {0, 1, 2, 3} since S_4 is four transitive on {0, 1, 2, 3}.

Note also that

$$t_0 t_1 t_2 t_1 t_2 = t_0 (0\ 3) t_2 t_1 \qquad \text{(by 3)}$$

$$= (0\ 3) t_3 t_2 t_1$$

$\Rightarrow \; N t_0 t_1 t_2 t_1 t_2 = N t_3 t_2 t_1$

$\Rightarrow$ [i j k j k] = [i j k] for i, j and k in {0, 1, 2, 3} since S_4 is four transitive on {0, 1, 2, 3}.

Similarly,

$t_0 t_1 t_2 t_3 t_2 = t_0 t_1 (0\ 1) t_3 t_2 t_3$ (by 3)

$= (0\ 1) t_1 t_0 t_3 t_2 t_3$

$\Rightarrow \; N t_0 t_1 t_2 t_3 t_2 = N t_1 t_0 t_3 t_2 t_3$

Then, the double coset [i j k l k] contains 12 distinct single cosets since each coset has two names.

By (1), we have

$N t_2 t_1 t_0 t_3 t_2 t_1 = N t_0 t_1 t_2 t_3 t_0$

$\Rightarrow$ [i j k l i j] = [i j k l i] for i, j, k and l in {0, 1, 2, 3} since S_4 is four transitive on {0, 1, 2, 3}.

By (2), we have

$N t_1 t_0 t_2 t_1 t_0 t_2 = N t_0 t_1 t_2 t_0 t_1$

$\Rightarrow$ [i j k i j k] = [i j k i j] for i, j, k and l in {0, 1, 2, 3} since S_4 is four transitive on {0, 1, 2, 3}.

Note that

$t_0 t_1 t_0 t_2 t_0 t_2 = t_0 t_1 (1\ 3) t_2 t_0$ (by 3)

$= (1\ 3) t_0 t_3 t_2 t_0$

$\Rightarrow \quad N t_0 t_1 t_0 t_2 t_0 t_2 = N t_0 t_3 t_2 t_0$

$\Rightarrow$ [i j i k i k] = [i j k i] for i, j and k in {0, 1, 2, 3} since S_4 is four transitive on {0, 1, 2, 3}.

Also,

$$t_0 t_1 t_2 t_0 t_1 t_0 = t_0 t_1 t_2 (2\,3) t_1 t_0 t_1 \qquad \text{(by 3)}$$

$$= (2\,3) t_0 t_1 t_3 t_1 t_0 t_1$$

$\Rightarrow \quad N t_0 t_1 t_2 t_0 t_1 t_0 = N t_0 t_1 t_3 t_1 t_0 t_1$

$\Rightarrow$ [i j k i j i] = [i j k j i j] for i, j and k in {0, 1, 2, 3} since S_4 is four transitive on {0, 1, 2, 3}.

And,

$$t_0 t_1 t_2 t_3 t_0 t_3 = t_0 t_1 t_2 (1\,2) t_0 t_3 t_0 \qquad \text{(by 3)}$$

$$= (1\,2) t_0 t_2 t_1 t_0 t_3 t_0$$

$\Rightarrow \quad N t_0 t_1 t_2 t_3 t_0 t_3 = N t_0 t_2 t_1 t_0 t_3 t_0$

$\Rightarrow$ [i j k l i l] = [i j k i l i] for i, j, k and l in {0, 1, 2, 3} since S_4 is four transitive on {0, 1, 2, 3}.

Note that

$$t_0 t_1 t_2 t_1 t_0 = t_0 (0\,3) t_2 t_1 t_2 t_0 \qquad \text{(by 3)}$$

$$= (0\,3) t_3 t_2 t_1 t_2 t_0$$

$\Rightarrow N t_0 t_1 t_2 t_1 t_0 = N t_3 t_2 t_1 t_2 t_0$

$\Rightarrow$ [i j k j i] = [i j k j l] for i, j and k in {0, 1, 2, 3}

since S_4 is four transitive on {0, 1, 2, 3}.

Using similar techniques, I have derived the relations:

$t_0 t_1 t_2 t_0 t_3 t_2 = (0\ 2)(1\ 3) t_1 t_2 t_3 t_1 t_0 t_3$,

$t_0 t_1 t_2 t_0 t_3 t_2 = (0\ 2)(1\ 3) t_3 t_0 t_1 t_3 t_2 t_1$,

and $t_0 t_1 t_2 t_0 t_3 t_2 = t_2 t_3 t_0 t_2 t_1 t_0$

$\Rightarrow N t_0 t_1 t_2 t_0 t_3 t_2 = N t_1 t_2 t_3 t_1 t_0 t_3 = N t_3 t_0 t_1 t_3 t_2 t_1 = N t_2 t_3 t_0 t_2 t_1 t_0$

Thus the double coset [i j k i l k] contains 6 distinct single cosets since each coset has four names.

Similarly,

$t_0 t_1 t_2 t_3 t_1 t_0 = (0\ 2)(1\ 3) t_3 t_0 t_2 t_1 t_0 t_3$,

and $t_0 t_1 t_2 t_3 t_1 t_0 = (0\ 3)(1\ 2) t_1 t_3 t_2 t_0 t_3 t_1$

$\Rightarrow N t_0 t_1 t_2 t_3 t_1 t_0 = N t_3 t_0 t_2 t_1 t_0 t_3 = N t_1 t_3 t_2 t_0 t_3 t_1$

Therefore, the double coset [i j k l j i] contains 8 distinct single cosets since each coset has three names.

Therefore, Cayley Graph of $L_2(23)$ over A_4 takes the form:

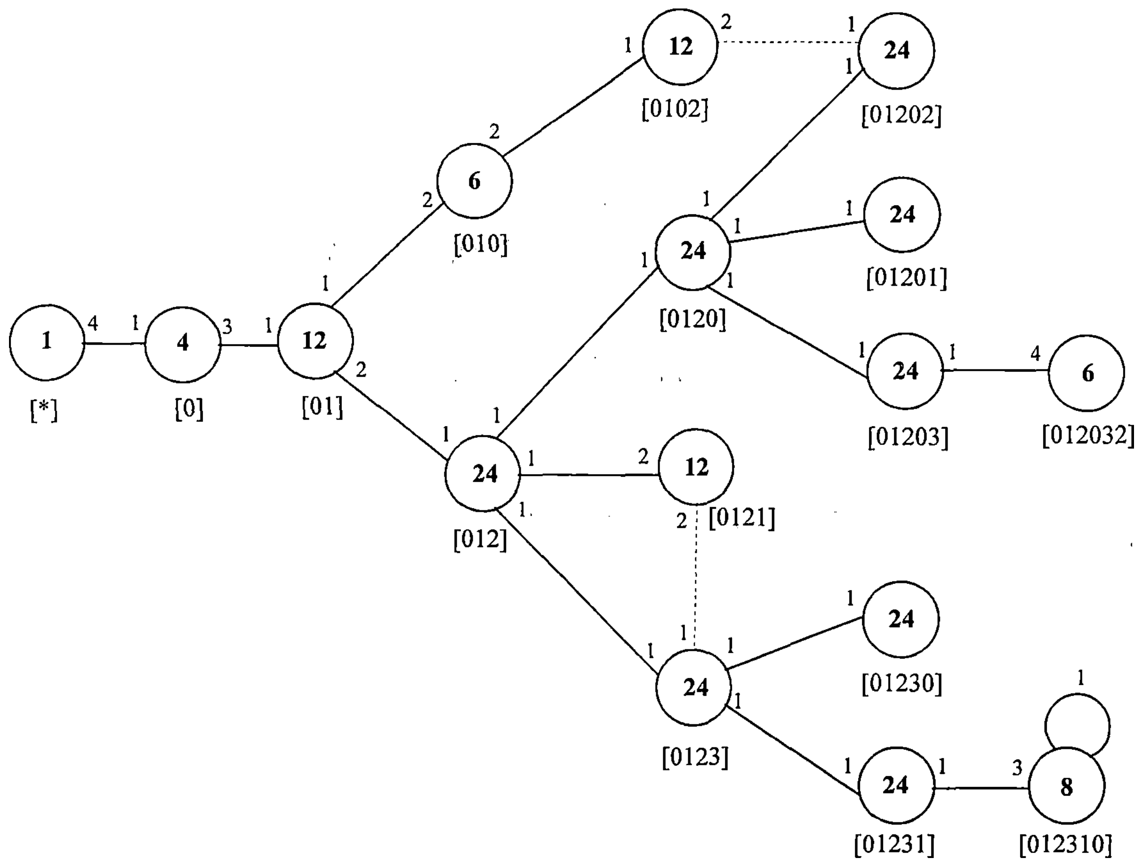

Figure 8. Cayley Graph of The Group $L_2(23)$ Over S_4

For the action of the four symmetric generators on the cosets of $L_2(23)$ over S_4, we label the 253 cosets as follows:

1 []	5 [2]
2 [4]	6 [1, 4]
3 [1]	7 [3, 4]
4 [3]	8 [2, 4]

9 [2, 1]

10 [4, 3]

11 [4, 1]

12 [2, 3]

13 [3, 1]

14 [1, 3]

15 [3, 2]

16 [2, 1, 4]

17 [4, 2]

18 [4, 3, 4]

19 [1, 2]

20 [4, 1, 4]

21 [2, 3, 4]

22 [3, 1, 4]

23 [1, 3, 4]

24 [3, 2, 4]

25 [3, 2, 1]

26 [1, 4, 3]

27 [4, 2, 4]

28 [2, 3, 2]

29 [1, 2, 4]

30 [1, 2, 1]

31 [3, 4, 1]

32 [1, 2, 3]

33 [4, 2, 1]

34 [2, 4, 3]

35 [2, 4, 1]

36 [4, 2, 3]

37 [4, 3, 1]

38 [2, 1, 3]

39 [4, 3, 2]

40 [2, 3, 1]

41 [3, 2, 1, 4]

42 [1, 4, 2]

43 [1, 4, 3, 4]

44 [1, 3, 1]

45 [2, 3, 2, 4]

46 [4, 1, 3]

47 [1, 2, 1, 4]

48 [4, 1, 2]

49 [2, 1, 4, 1]

50 [1, 3, 2]

51 [1, 2, 3, 4]

52 [4, 2, 1, 4]

53 [3, 4, 2]

54 [1, 3, 4, 3]

55 [3, 1, 2]

56 [2, 4, 1, 4]

57 [4, 2, 3, 4]

58 [4, 3, 1, 4]

59 [2, 1, 3, 4]

60 [4, 3, 2, 4]

61 [2, 3, 1, 4]

62 [4, 3, 2, 1]

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where

$t_0 =$ (1, 2)(3, 6)(4, 7)(5, 8)(9, 16)(10, 18)(11, 20)(12, 21)(13, 22)(14, 23)(15, 24)(17, 27)(19, 29)(25, 41)(26, 43)(28, 45)(30, 47)(31, 49)(32, 51)(33, 52)(34, 54)(35, 56)(36, 57)(37, 58)(38, 59)(39, 60)(40, 61)(42, 64)(44, 66)(46, 69)(48, 72)(50, 74)(53, 79)(55, 81)(62, 95)(63, 73)(65, 75)(67, 101)(71, 106)(76, 114)(77, 116)(78, 118)(80, 87)(82, 92)(83, 121)(84, 123)(85, 124)(86, 126)(88, 128)(89, 129)(90, 131)(91, 132)(93, 134)(94, 135)(96, 109)(97, 133)(98, 138)(99, 139)(100, 140)(103, 142)(104, 143)(105, 144)(107, 146)(108, 147)(110, 148)(111, 149)(112, 150)(113, 151)(115, 154)(117, 157)(119, 159)(120, 160)(122, 162)(125, 167)(127, 169)(130, 174)(136, 153)(137, 161)(141, 156)(145, 163)(152, 158)(155, 212)(164, 219)(166, 204)(168, 175)(170, 205)(171, 227)(172, 229)(173, 215)(176, 233)(177, 235)(178, 224)(179, 192)(180, 238)(181, 214)(182, 186)(183, 240)(184, 208)(185, 194)(187, 201)(188, 239)(189, 241)(190, 213)(191, 196)(195, 242)(197, 199)(198, 222)(200, 202)(203, 245)(206, 244)(207, 236)(209, 217)(211, 246)(218, 247)(220, 248)(221, 232)(223, 225)(226, 249)(230, 234)(231, 251)(237, 253)(250, 252)

$t_1 =$ (1, 3)(2, 11)(4, 13)(5, 9)(6, 20)(7, 31)(8, 35)(10, 37)(12, 40)(14, 44)(15, 25)(16, 49)(17, 33)(18, 67)(19, 30)(21, 75)(22, 56)(23, 83)(24, 87)(26, 89)(27, 98)(28, 70)(29, 103)(32, 107)(34, 110)(36, 94)(38, 97)(39, 62)(41, 65)(42, 85)(43, 112)(46, 109)(47, 121)(48, 73)(50, 77)(51, 152)(52, 101)(53, 91)(54, 127)(55, 82)(57, 163)(58,138)(59, 170)(60, 175)(61, 80)(63, 179)(64, 119)(66, 142)(68, 188)(69, 192)(71, 190)(72, 197)(74, 201)(76, 203)(78, 209)(79, 111)(81, 214)(84, 216)(86, 218)(88, 184)(90, 189)(92, 205)(93, 136)(95, 145)(96, 199)(100, 141)(102, 186)(104, 212)(106, 220)(108, 194)(113, 177)(114, 183)(115, 155)(116, 158)(117, 172)(118, 207)(120, 161)(122, 165)(123, 162)(124, 150)(125, 195)(126, 232)(128, 238)(129, 159)(130, 191)(131, 233)(132, 169)(133, 181)(134, 253)(135, 168)(137, 204)(139, 244)(140, 231)(144, 206)(146, 187)(147, 166)(148, 149)(151, 202)(153, 180)(156, 248)(157, 224)(160, 223)(164, 247)(167, 219)(171, 210)(174, 246)(176, 196)(178, 217)(182, 239)(185, 225)(193, 228)(200, 245)(208, 237)(211, 241)(213, 251)(215, 250)(221, 242)(226, 243)(227, 249)(229, 236)(234, 252)(235, 240)

$t_2 =$ (1, 5)(2, 17)(3, 19)(4, 15)(6, 42)(7, 53)(8, 27)(9, 30)(10, 39)(11, 48)(12, 28)(13, 55)(14, 50)(16, 117)(18, 105)(20, 100)(21, 125)(22, 96)(23, 133)(24, 64)(25, 73)(26, 93)(29, 79)(31, 112)(32, 65)(33, 82)(34, 115)(35, 120)(36, 80)(37, 127)(38, 122)(40, 130)(41, 236)(43, 62)(44,102)(45, 157)(46, 113)(47, 167)(49, 76)(51, 240)(52, 223)(54, 91)(56, 88)(57, 230)(58, 213)(59, 239)(60, 140)(61, 185)(63, 183)(68, 158)(69, 241)(70, 161)(72, 144)(74, 109)(75, 207)(77, 141)(78, 211)(81, 97)(83, 220)(84, 145)(85, 186)(86, 222)(87, 225)(89, 232)(90, 228)(92, 234)(94, 237)(95, 150)(98, 191)(99, 168)(101, 153)(103, 229)(104, 187)(106, 136)(107, 243)(108, 221)(110, 244)(111, 226)(114, 134)(116, 247)(118, 155)(119, 208)(121, 164)(123, 251)(124, 197)(126, 181)(128, 154)(131, 146)(132, 160)(135, 224)(137, 152)(138, 176)(139, 195)(142, 202)(143, 173)(148, 235)(149, 205)(151, 190)(156, 231)(159, 227)(162, 194)(163, 218)(165, 203)(169, 189)(170, 253)(171, 206)(172, 214)(174, 188)(175, 233)(177, 184)(178, 249)(179, 250)(180, 193)(182, 215)(196, 242)(198, 200)(199, 201)(204, 252)(210, 216)(212, 248)(219, 246)(238, 245)

$t_3 =$ (1, 4)(2, 10)(3, 14)(5, 12)(6, 26)(7, 18)(8, 34)(9, 38)(11, 46)(13, 44)(15, 28)(16, 63)(17, 36)(19, 32)(20, 71)(21, 43)(22, 78)(23, 54)(24, 86)(25, 90)(27, 99)(29, 92)(30, 68)(31, 108)(33, 111)(35, 119)(37, 97)(39, 65)(40, 109)(41, 137)(42, 88)(45, 118)(48, 76)(49, 93)(50, 80)(51, 73)(52, 156)(53, 104)(55, 84)(56, 113)(57, 106)(58, 166)(59, 82)(60, 173)(61, 178)(62, 180)(64, 94)(66, 126)(67, 189)(69, 139)(70, 194)(72, 196)(74, 200)(75, 204)(77, 210)(79, 110)(81, 182)(83, 217)(85, 219)(87, 224)(89, 188)(91, 206)(95, 207)(96, 202)(101, 211)(102, 187)(103, 231)(105, 145)(107, 190)(112, 153)(114, 141)(115, 158)(116, 174)(117, 176)(120, 164)(121, 183)(122, 168)(123, 161)(125, 198)(127, 171)(128, 151)(129, 192)(130, 193)(131, 186)(132, 245)(133, 215)(134, 143)(135, 159)(136, 203)(138, 208)(140, 184)(142, 221)(144, 218)(146, 242)(147, 148)(149, 191)(150, 249)(152, 179)(154, 230)(155, 251)(157, 212)(162, 248)(163, 246)(165, 243)(167, 205)(169, 181)(170, 209)(175, 195)(177, 216)(185, 201)(199, 252)(213, 220)(214, 238)(222, 240)(225, 250)(226, 244)(227, 236)(228, 237)(232, 233)(234, 239)(235, 253)(241, 247)

Proof of Isomorphism

We obtained the collapsed Cayley graph from the information contained in the following computations

$$\frac{|N|}{|N|}+\frac{|N|}{|N^{(0)}|}+\frac{|N|}{|N^{(01)}|}+\frac{|N|}{|N^{(010)}|}+\frac{|N|}{|N^{(012)}|}+\frac{|N|}{|N^{(0102)}|}+\frac{|N|}{|N^{(0120)}|}+\frac{|N|}{|N^{(0121)}|}+\frac{|N|}{|N^{(0123)}|}$$
$$+\frac{|N|}{|N^{(01201)}|}+\frac{|N|}{|N^{(01202)}|}+\frac{|N|}{|N^{(01203)}|}+\frac{|N|}{|N^{(01230)}|}+\frac{|N|}{|N^{(01231)}|}+\frac{|N|}{|N^{(012032)}|}+\frac{|N|}{|N^{(012310)}|}$$

$$=\frac{24}{24}+\frac{24}{6}+\frac{24}{2}+\frac{24}{4}+\frac{24}{1}+\frac{24}{2}+\frac{24}{1}+\frac{24}{2}+\frac{24}{1}+\frac{24}{1}$$
$$+\frac{24}{1}+\frac{24}{1}+\frac{24}{1}+\frac{24}{1}+\frac{24}{4}+\frac{24}{3}$$

$$= 1 + 4 + 12 + 6 + 24 + 12 + 24 + 12 + 24 + 24 + 24 + 24 + 24 + 24 + 6 + 8 = 253 .$$

This tells us that the maximum possible index of N in G is 253. It follows that the order of the image group G is at most $|N| * 253 = 24 * 253 = 6072$. The order of G can be established by considering G as a permutation group on the 253 cosets that we have found.

We know that the action of the control group N on the cosets is well-defined and the actions of the symmetric generators on the cosets are well-defined as well.

We will show that:

(i) t_0 has just four images under conjugation by N.

(ii) The three additional relations hold within the symmetric group S_{253}.

However, we can verify that the image is the projective special linear group $L_2(23)$:

We first define the linear fractional permutations of the projective line $L_2(23)$ as follows

$$x: \left(\tau \mapsto \frac{10\tau - 16}{\tau - 1}\right) \equiv (1, 24, 10, 17)(2, 4, 8, 19)(3, 7, 9, 15)(5, 20, 23, 16)(6, 18, 11, 14)(12, 22, 13, 21)$$

$$y: \left(\tau \mapsto \frac{-3\tau + 11}{\tau + 3}\right) \equiv (1, 2)(3, 8)(4, 13)(5, 11)(6, 12)(7, 22)(9, 14)(10, 18)(15, 16)(17, 21)(19, 23)(20, 24)$$

$t_0: \left(\tau \mapsto \frac{10\tau + 3}{\tau - 10}\right) \equiv (1, 19)\,(2, 23)\,(3, 15)\,(4, 12)\,(5, 17)\,(6, 13)\,(7, 14)\,(8, 16)\,(9, 22)$
$(10, 24)\,(11, 21)\,(18, 20)$

We note that xy has order 3, and hence $N = \langle x, y\rangle \cong S_4$.

The permutation t_0 has just four images under conjugation by N, namely:

$t_0 = (1, 19)(2, 23)(3, 15)(4, 12)(5, 17)(6, 13)(7, 14)(8, 16)(9, 22)(10, 24)(11, 21)(18, 20)$

$t_1 = (1, 20)(2, 24)(3, 7)(4, 16)(5, 19)(6, 9)(8, 22)(10, 17)(11, 23)(12, 14)(13, 15)(18, 21)$

$t_2 = (1, 17)(2, 20)(3, 21)(4, 10)(5, 8)(6, 22)(7, 9)(11, 12)(13, 19)(14, 16)(15, 18)(23, 24)$

$t_3 = (1, 24)(2, 21)(3, 11)(4, 23)(5, 6)(7, 12)(8, 17)(9, 15)(10, 16)(13, 18)(14, 22)(19, 20)$

And the action of x and y, and hence N, on these symmetric generators is as follows:

$x: (t_0, t_1, t_2, t_3)$

$y: (t_2, t_3)$

Now we check our relations; that is

$$\begin{aligned} t_0^{t_0 t_1 t_2 t_3 t_0 t_1 t_2 t_3 t_0 t_1 t_2} &= (1, 24)\,(2, 21)\,(3, 11)\,(4, 23)\,(5, 6)\,(7, 12)\,(8, 17) \\ &\quad (9, 15)\,(10, 16)\,(13, 18)\,(14, 22)\,(19, 20) \\ &= t_3 \end{aligned}$$

$$\begin{aligned} t_1^{t_0 t_1 t_2 t_3 t_0 t_1 t_2 t_3 t_0 t_1 t_2} &= (1, 19)\,(2, 23)\,(3, 15)\,(4, 12)\,(5, 17)\,(6, 13)\,(7, 14) \\ &\quad (8, 16)\,(9, 22)\,(10, 24)\,(11, 21)\,(18, 20) \\ &= t_0 \end{aligned}$$

$$t_2^{t_0 t_1 t_2 t_3 t_0 t_1 t_2 t_3 t_0 t_1 t_2} = (1, 20)(2, 24)(3, 7)(4, 16)(5, 19)(6, 9)(8, 22)(10, 17)(11, 23)(12, 14)(13, 15)(18, 21) = t_1$$

$$t_3^{t_0 t_1 t_2 t_3 t_0 t_1 t_2 t_3 t_0 t_1 t_2} = (1, 17)(2, 20)(3, 21)(4, 10)(5, 8)(6, 22)(7, 9)(11, 12)(13, 19)(14, 16)(15, 18)(23, 24) = t_2$$

This means that $t_0\, t_1\, t_2\, t_3\, t_0\, t_1\, t_2\, t_3\, t_0\, t_1\, t_2$ acts as the permutation (0 3 2 1) on the symmetric generators, that is

$t_0\, t_1\, t_2\, t_3\, t_0\, t_1\, t_2\, t_3\, t_0\, t_1\, t_2 = (0\,3\,2\,1) = x^3$, hence our first relation holds in $L_2(23)$.

Similarly,

$$t_0^{t_0 t_1 t_2 t_0 t_1 t_2 t_0 t_1 t_2 t_0 t_1} = (1, 17)(2, 20)(3, 21)(4, 10)(5, 8)(6, 22)(7, 9)(11, 12)(13, 19)(14, 16)(15, 18)(23, 24) = t_2$$

$$t_1^{t_0 t_1 t_2 t_0 t_1 t_2 t_0 t_1 t_2 t_0 t_1} = (1, 19)(2, 23)(3, 15)(4, 12)(5, 17)(6, 13)(7, 14)(8, 16)(9, 22)(10, 24)(11, 21)(18, 20) = t_0$$

$$t_2^{t_0 t_1 t_2 t_0 t_1 t_2 t_0 t_1 t_2 t_0 t_1} = (1, 20)(2, 24)(3, 7)(4, 16)(5, 19)(6, 9)(8, 22)(10, 17)(11, 23)(12, 14)(13, 15)(18, 21) = t_1$$

$$t_3^{t_0 t_1 t_2 t_0 t_1 t_2 t_0 t_1 t_2 t_0 t_1} = (1, 24)(2, 21)(3, 11)(4, 23)(5, 6)(7, 12)(8, 17)(9, 15)(10, 16)(13, 18)(14, 22)(19, 20) = t_3$$

Thus $t_1\, t_0\, t_2\, t_1\, t_0\, t_2\, t_1\, t_0\, t_2\, t_1\, t_0$ acts as the permutation (0 2 1) on the symmetric generators, that is

$t_1\, t_0\, t_2\, t_1\, t_0\, t_2\, t_1\, t_0\, t_2\, t_1\, t_0 = (0\ 2\ 1) = \pi^2$, which proves relation (2).

Also,

$$\begin{aligned} t_0^{(t_0\, t_1)^3} &= (1, 19)\,(2, 23)\,(3, 15)\,(4, 12)\,(5, 17)\,(6, 13)\,(7, 14)\,(8, 16) \\ &\quad (9, 22)\,(10, 24)\,(11, 21)\,(18, 20) \\ &= t_0 \end{aligned}$$

$$\begin{aligned} t_1^{(t_0\, t_1)^3} &= (1, 20)\,(2, 24)\,(3, 7)\,(4, 16)\,(5, 19)\,(6, 9)\,(8, 22)\,(10, 17) \\ &\quad (11, 23)\,(12, 14)\,(13, 15)\,(18, 21) \\ &= t_1 \end{aligned}$$

$$\begin{aligned} t_2^{(t_0\, t_1)^3} &= (1, 24)\,(2, 21)\,(3, 11)\,(4, 23)\,(5, 6)\,(7, 12)\,(8, 17)\,(9, 15) \\ &\quad (10, 16)\,(13, 18)\,(14, 22)\,(19, 20) \\ &= t_3 \end{aligned}$$

$$\begin{aligned} t_3^{(t_0\, t_1)^3} &= (1, 17)\,(2, 20)\,(3, 21)\,(4, 10)\,(5, 8)\,(6, 22)\,(7, 9)\,(11, 12) \\ &\quad (13, 19)\,(14, 16)\,(15, 18)\,(23, 24) \\ &= t_2 \end{aligned}$$

That is, $(t_0\, t_1)^3$ acts as the transposition (2 3) on the symmetric generators, that is $(t_0\, t_1)^3 = y$, and that proves our third relation.

N is maximal in $L_2(23)$ and $t_0 \notin N$, hence N and t_0 generate the whole group, $L_2(23)$, and so $L_2(23)$ is an image of G. Thus $|G| \geq |L_2(23)|$.

Now since we have that $|G| \leq 6072 = |L_2(23)|$.

Therefore, $|G| \leq 6072 = |L_2(23)| \leq |G|$,

which proves the isomorphism, that is $G \cong L_2(23)$.

CHAPTER NINE

THE GROUP $PGL_2(17)$ OVER A_4

A presentations of the group $2^{*4}:A_4$ is given by:

$<x, y, t \mid x^3, y^3, (x\,y)^2, t^2, (t, x)>$

where the action of x and y on the symmetric generators is given by:

x = $(1\,2\,3)$,

y = $(0\,1\,2)$,

and $N^0 = <x>$.

We factor the progenitor by the following relations

$(y\,t_0)^{18} = 1$, $(x\,y\,t_0)^{16} = 1$ and $(y\,t_0\,t_3)^4 = 1$

To obtain the group G,

$$G \cong \frac{2^{*4}:A_4}{(y\,t_0)^{18}, (x\,y\,t_0)^{16}, (y\,t_0\,t_3)^4}.$$

Then, the index of N in G is 408.

Manual Double Coset Enumeration

The relation $(y\,t_0)^{18} = 1$

$\Rightarrow$ $yt_0\,yt_0\,yt_0\,yt_0\,yt_0\,yt_0\,yt_0\,yt_0\,yt_0\,yt_0\,yt_0\,yt_0\,yt_0\,yt_0\,yt_0\,yt_0\,yt_0\,yt_0 = 1$

$$\Rightarrow \begin{array}{l} y^3 y^{-2} t_0 y^2 y^{-1} t_0 y t_0 y^3 y^{-2} t_0 y^2 y^{-1} t_0 y t_0 y^3 y^{-2} t_0 y^2 y^{-1} t_0 y t_0 y^3 y^{-2} t_0 \\ y^2 y^{-1} t_0 y t_0 y^3 y^{-2} t_0 y^2 y^{-1} t_0 y t_0 y^3 y^{-2} t_0 y^2 y^{-1} t_0 y t_0 = 1 \end{array}$$

$$\Rightarrow t_0^{y^2} t_0^{y} t_0 t_0^{y^2} t_0^{y} t_0 t_0^{y^2} t_0^{y} t_0 t_0^{y^2} t_0^{y} t_0 t_0^{y^2} t_0^{y} t_0 t_0^{y^2} t_0^{y} t_0 = 1$$

$$\Rightarrow t_2 t_1 t_0 t_2 t_1 t_0 t_2 t_1 t_0 t_2 t_1 t_0 t_2 t_1 t_0 t_2 t_1 t_0 = 1$$

$$\Rightarrow t_2 t_1 t_0 t_2 t_1 t_0 t_2 t_1 t_0 = t_0 t_1 t_2 t_0 t_1 t_2 t_0 t_1 t_2 \qquad (1)$$

$$\Rightarrow N t_2 t_1 t_0 t_2 t_1 t_0 t_2 t_1 t_0 = N t_0 t_1 t_2 t_0 t_1 t_2 t_0 t_1 t_2$$

The relation $(x y t_0)^{16} = 1$ with $x y = \pi$ becomes

$$(\pi t_0)^{16} = 1$$

$$\Rightarrow \pi t_0 \pi t_0 \pi t_0 \pi t_0 \pi t_0 \pi t_0 \pi t_0 \pi t_0 \pi t_0 \pi t_0 \pi t_0 \pi t_0 \pi t_0 \pi t_0 \pi t_0 \pi t_0 = 1$$

$$\Rightarrow t_0^{\pi} t_0 t_0^{\pi} t_0 t_0^{\pi} t_0 t_0^{\pi} t_0 t_0^{\pi} t_0 t_0^{\pi} t_0 t_0^{\pi} t_0 t_0^{\pi} t_0 = 1$$

(since $\pi = (0\ 1)(2\ 3)$, so $\pi = \pi^{-1}$)

$$\Rightarrow t_1 t_0 t_1 t_0 t_1 t_0 t_1 t_0 t_1 t_0 t_1 t_0 t_1 t_0 t_1 t_0 = 1$$

$$\Rightarrow t_1 t_0 t_1 t_0 t_1 t_0 t_1 t_0 = t_0 t_1 t_0 t_1 t_0 t_1 t_0 t_1 \qquad (2)$$

$$\Rightarrow N t_1 t_0 t_1 t_0 t_1 t_0 t_1 t_0 = N t_0 t_1 t_0 t_1 t_0 t_1 t_0 t_1$$

The relation $(y t_0 t_3)^4 = 1$

$$\Rightarrow y t_0 t_3 y t_0 t_3 y t_0 t_3 y t_0 t_3 = 1$$

$$\Rightarrow y t_0 t_3 y^3 y^{-2} t_0 t_3 y^2 y^{-1} t_0 t_3 y t_0 t_3 = 1$$

$$\Rightarrow y t_0 t_3 t_0^{y^2} t_3^{y^2} t_0^{y} t_3^{y} t_0 t_3 = 1$$

$$\Rightarrow (0\ 1\ 2) t_0 t_3 t_2 t_3 t_1 t_3 t_0 t_3 = 1$$

$\Rightarrow \ (0\ 1\ 2) = t_3\ t_0\ t_3\ t_1\ t_3\ t_2\ t_3\ t_0$ (3)

$\Rightarrow \ (0\ 1\ 2)\, t_0\ t_3\ t_2\ t_3 = t_3\ t_0\ t_3\ t_1$

$\Rightarrow \ N\, t_0\ t_3\ t_2\ t_3 = N\, t_3\ t_0\ t_3\ t_1$

Thus, [i j l j] = [i j i l] for i, j, and l in {0,1,2,3}.

Using (3), we can write

$(0\ 1\ 2)\, t_0\ t_3\ t_2 = t_3\ t_0\ t_3\ t_1\ t_3$

$\Rightarrow \ N\, t_0\ t_3\ t_2 = N\, t_3\ t_0\ t_3\ t_1\ t_3$

Thus, [i j i l i] = [i j l] for i, j, k, and l in {0,1,2,3}.

Also, by (3), we can write

$(0\ 1\ 2)\, t_0\ t_3\ t_2\ t_3\ t_1 = t_3\ t_0\ t_3$

$\Rightarrow \ N\, t_0\ t_3\ t_2\ t_3\ t_1 = N\, t_3\ t_0\ t_3$

Thus, [i j l j k] = [i j i] for i, j, k, and l in {0,1,2,3}.

Using (3), we can also write

$(0\ 1\ 2)\, t_0\ t_3\ t_2\ t_3\ t_2 = t_3\ t_0\ t_3\ t_1\ t_2$

$\Rightarrow \ N\, t_0\ t_3\ t_2\ t_3\ t_2 = N\, t_3\ t_0\ t_3\ t_1\ t_2$

Thus, [i j l j l] = [i j i l k] for i, j, k, and l in {0,1,2,3}.

Note that,

$$t_0\ t_1\ t_0\ t_1\ t_2 = t_0\ (0\ 2\ 3)\ t_0\ t_1\ t_3\ t_1 \qquad \text{(by 3)}$$

$$= (0\ 2\ 3)\ t_2\ t_0\ t_1\ t_3\ t_1$$

$$\Rightarrow\ N t_0\ t_1\ t_0\ t_1\ t_2 = N t_2\ t_0\ t_1\ t_3\ t_1$$

Thus, [i j i j k] = [i j k l k] for i, j, k, and l in {0,1,2,3}.

Similarly,

$$t_0\ t_1\ t_0\ t_2\ t_0 = t_0\ (1\ 2\ 3)\ (1\ 3\ 2)\ t_1\ t_0\ t_2\ t_0$$

$$= t_0\ (1\ 2\ 3) t_0\ t_1\ t_0\ t_3 \qquad \text{(by 3)}$$

$$= (1\ 2\ 3)\ t_1\ t_0\ t_3$$

$$\Rightarrow\ N t_0\ t_1\ t_0\ t_2\ t_0 = N t_1\ t_0\ t_3$$

Thus, [i j i k i] = [i j k] for i, j, and k in {0,1,2,3},

This gives us also:

$$N t_0\ t_1\ t_0\ t_2 = N t_1\ t_0\ t_3\ t_0$$

Then [i j i k] = [i j k j] for i, j, and k in {0,1,2,3}.

{0,1,2,3}.

Also, we can write

$$t_0\ t_1\ t_0\ t_2\ t_1\ t_0 = (1\ 2\ 3)\ t_1\ t_0\ t_3\ t_0\ t_1\ t_0$$

$$= (1\ 2\ 3)\ t_1\ t_0\ (1\ 2\ 3)\ t_0\ t_3\ t_0\ t_2 \qquad \text{(by 3)}$$

$$= (1\ 3\ 2)\ t_2\ t_0\ t_0\ t_3\ t_0\ t_2$$

$= (1\ 3\ 2)\, t_2\ t_3\ t_0\ t_2$

$\Rightarrow\ N t_0\ t_1\ t_0\ t_2\ t_1\ t_0 = N t_2\ t_3\ t_0\ t_2$

Thus, [i j i k j i] = [i j k i] for i, j, and k in {0,1,2,3}.

Using the same techniques, I have also found the following relations:

$t_0\ t_1\ t_2\ t_0\ t_1 = (0\ 1)\,(2\ 3)\, t_2\ t_3\ t_0\ t_2\ t_3$

$\Rightarrow\ N t_0\ t_1\ t_2\ t_0\ t_1 = N t_2\ t_3\ t_0\ t_2\ t_3$

Then the double coset [i j k i j] contains 6 distinct single cosets because every coset has two names.

Similarly,

$t_0\ t_1\ t_2\ t_3\ t_0 = (0\ 1)\,(2\ 3)\, t_2\ t_0\ t_1\ t_3\ t_2$,

and $t_0\ t_1\ t_2\ t_3\ t_0 = (0\ 3)\,(1\ 2)\, t_1\ t_2\ t_0\ t_3\ t_1$

$\Rightarrow\ N t_0\ t_1\ t_2\ t_3\ t_0 = N t_2\ t_0\ t_1\ t_3\ t_2 = N t_1\ t_2\ t_0\ t_3\ t_1$

Then the double coset [i j k l i] contains 4 distinct single cosets since each coset has three names.

Also,

$t_0\ t_1\ t_3\ t_2\ t_0 = (0\ 1)\,(2\ 3)\, t_3\ t_0\ t_1\ t_2\ t_3$,

and $t_0\ t_1\ t_3\ t_2\ t_0 = (0\ 2)\,(1\ 3)\, t_1\ t_3\ t_0\ t_2\ t_1$

$\Rightarrow\ N t_0\ t_1\ t_3\ t_2\ t_0 = N t_3\ t_0\ t_1\ t_2\ t_3 = N t_1\ t_3\ t_0\ t_2\ t_1$

Then the double coset [i j l k i] contains 4 distinct single cosets because every coset has three names.

$t_0\, t_1\, t_0\, t_1\, t_3\, t_2 = (0\ 3)\,(1\ 2)\, t_1\, t_0\, t_1\, t_0\, t_2\, t_3$

$\Rightarrow\ N t_0\, t_1\, t_0\, t_1\, t_3\, t_2 = N t_1\, t_0\, t_1\, t_0\, t_2\, t_3$

Then the double coset [i j i j l k] contains 6 distinct single cosets since each coset has two names.

$t_0\, t_1\, t_0\, t_2\, t_3\, t_1 = (0\ 1)\,(2\ 3)\, t_1\, t_2\, t_1\, t_0\, t_3\, t_2$,

and $t_0\, t_1\, t_0\, t_2\, t_3\, t_1 = (0\ 2)\,(1\ 3)\, t_2\, t_0\, t_2\, t_1\, t_3\, t_0$

$\Rightarrow\ N t_0\, t_1\, t_0\, t_2\, t_3\, t_1 = N t_1\, t_2\, t_1\, t_0\, t_3\, t_2 = N t_2\, t_0\, t_2\, t_1\, t_3\, t_0$

Then the double coset [i j i k l j] contains 4 distinct single cosets because every coset has three names.

$t_0\, t_1\, t_0\, t_2\, t_3\, t_2 = (0\ 2\ 3)\, t_0\, t_2\, t_0\, t_3\, t_1\, t_3$,

and $t_0\, t_1\, t_0\, t_2\, t_3\, t_2 = (0\ 2\ 3)\, t_0\, t_2\, t_0\, t_3\, t_1\, t_3$

$\Rightarrow\ N t_0\, t_1\, t_0\, t_2\, t_3\, t_2 = N t_0\, t_2\, t_0\, t_3\, t_1\, t_3 = N t_0\, t_2\, t_0\, t_3\, t_1\, t_3$

Then the double coset [i j i k l k] contains 4 distinct single cosets since each coset has three names.

$t_0\, t_1\, t_0\, t_3\, t_2\, t_1 = (0\ 3)\,(1\ 2)\, t_3\, t_0\, t_3\, t_1\, t_2\, t_0$, and $t_0\ t_1\ t_0\ t_3\ t_2\ t_1 = (0\ 1)(2\ 3)\ t_1\ t_3\ t_1\ t_0\ t_2\ t_3$

$\Rightarrow\ N t_0\ t_1\ t_0\ t_3\ t_2\ t_1 = N t_3\ t_0\ t_3\ t_1\ t_2\ t_0 = N t_1\ t_3\ t_1\ t_0\ t_2\ t_3$

Then the double coset [i j i l k j] contains 4 distinct single cosets since each coset has three names.

$t_0\ t_1\ t_0\ t_3\ t_2\ t_3 = (0\ 3\ 1)\ t_0\ t_2\ t_0\ t_1\ t_3\ t_1$

and $t_0\ t_1\ t_0\ t_3\ t_2\ t_3 = (0\ 3\ 2)\ t_0\ t_3\ t_0\ t_2\ t_1\ t_2$

$\Rightarrow\ N t_0\ t_1\ t_0\ t_3\ t_2\ t_3 = N t_0\ t_2\ t_0\ t_1\ t_3\ t_1 = N t_0\ t_3\ t_0\ t_2\ t_1\ t_2$

Then the double coset [i j i l k l] contains 4 distinct single cosets since each coset has three names.

$t_0\ t_1\ t_0\ t_1\ t_0\ t_1\ t_0 = (0\ 3\ 2)\ t_2\ t_1\ t_2\ t_1\ t_2\ t_1\ t_2$,

and $t_0\ t_1\ t_0\ t_1\ t_0\ t_1\ t_0 = (0\ 2\ 3)\ t_3\ t_1\ t_3\ t_1\ t_3\ t_1\ t_3$

$\Rightarrow\ N t_0\ t_1\ t_0\ t_1\ t_0\ t_1\ t_0 = N t_2\ t_1\ t_2\ t_1\ t_2\ t_1\ t_2 = N t_3\ t_1\ t_3\ t_1\ t_3\ t_1\ t_3 \quad (*)$

Then the double coset [i j i j i j i] contains 4 distinct single cosets because every coset has three names.

$t_0\ t_1\ t_0\ t_1\ t_3\ t_0\ t_3 = (1\ 2\ 3)\ t_3\ t_0\ t_3\ t_0\ t_1\ t_3\ t_1$,

and $t_0\ t_1\ t_0\ t_1\ t_3\ t_0\ t_3 = (0\ 2\ 3)\ t_1\ t_3\ t_1\ t_3\ t_0\ t_1\ t_0$

$\Rightarrow\ N t_0\ t_1\ t_0\ t_1\ t_3\ t_0\ t_3 = N t_3\ t_0\ t_3\ t_0\ t_1\ t_3\ t_1 = N t_1\ t_3\ t_1\ t_3\ t_0\ t_1\ t_0$

Then the double coset [i j i j l i l] contains 4 distinct single cosets since each coset has three names.

$t_0\, t_1\, t_0\, t_2\, t_1\, t_2\, t_1 = (0\ 3\ 1)\, t_0\, t_2\, t_0\, t_3\, t_2\, t_3\, t_2,$

and $t_0\, t_1\, t_0\, t_2\, t_1\, t_2\, t_1 = (0\ 3\ 2)\, t_0\, t_3\, t_0\, t_1\, t_3\, t_1\, t_3$

$\Rightarrow\ N t_0\, t_1\, t_0\, t_2\, t_1\, t_2\, t_1 = N t_0\, t_2\, t_0\, t_3\, t_2\, t_3\, t_2 = N t_0\, t_3\, t_0\, t_1\, t_3\, t_1\, t_3$

Then the double coset [i j i k j k j] contains 4 distinct single cosets since each coset has three names.

$t_0\, t_1\, t_0\, t_2\, t_1\, t_3\, t_2 = (0\ 3)\,(1\ 2)\, t_1\, t_0\, t_1\, t_3\, t_0\, t_2\, t_3$

$\Rightarrow\ N t_0\, t_1\, t_0\, t_2\, t_1\, t_3\, t_2 = N t_1\, t_0\, t_1\, t_3\, t_0\, t_2\, t_3$

Then the double coset [i j i k j l k] contains 6 distinct single cosets since each coset has two names.

Now,

$t_0\, t_1\, t_0\, t_1\, t_0\, t_1\, t_0\, t_1 = (0\ 1)\,(2\ 3)\, t_2\, t_3\, t_2\, t_3\, t_2\, t_3\, t_2\, t_3$

$\Rightarrow\ N t_0\, t_1\, t_0\, t_1\, t_0\, t_1\, t_0\, t_1 = N t_2\, t_3\, t_2\, t_3\, t_2\, t_3\, t_2\, t_3$

Also, by (*), we can write

$N t_0\, t_1\, t_0\, t_1\, t_0\, t_1\, t_0\, t_1 = N t_2\, t_1\, t_2\, t_1\, t_2\, t_1\, t_2\, t_1 = N t_3\, t_1\, t_3\, t_1\, t_3\, t_1\, t_3\, t_1$

And, using (2), we conclude that the double coset [i j i j i j i j] consists of a single coset.

Cayley graph of the group is shown below:

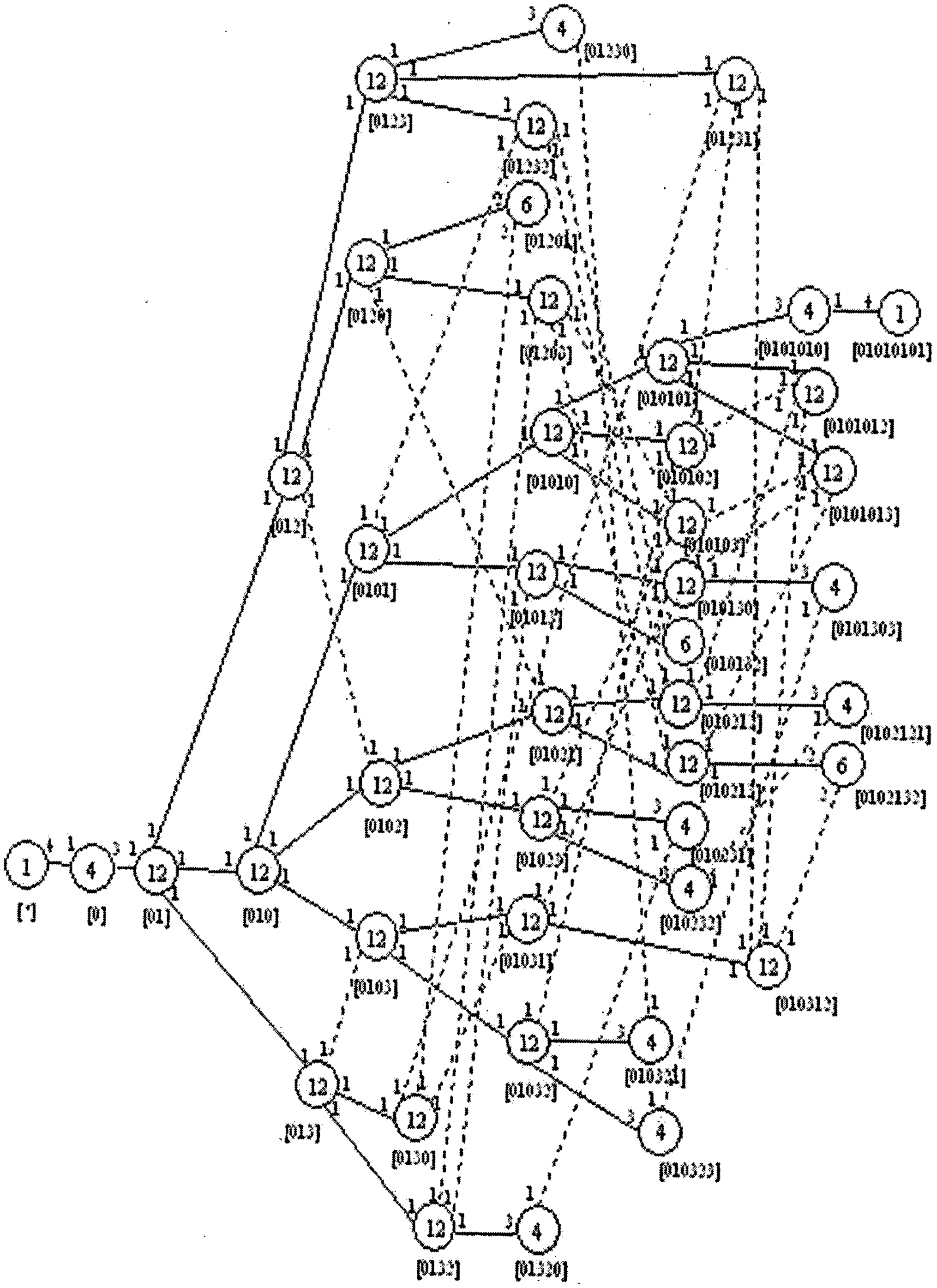

Figure 9. Cayley Graph of The Group $PGL_2(17)$ Over A_4

For the action of the four symmetric generators on the cosets of $PGL_2(17)$ over A_4, we label the 408 cosets as follows:

1 []	20 [2, 3]
2 [4]	21 [1, 2, 4]
3 [1]	22 [3, 2, 4]
4 [2]	23 [4, 2, 1]
5 [1, 4]	24 [1, 3, 4]
6 [3]	25 [1, 4, 2]
7 [2, 4]	26 [3, 1, 4]
8 [2, 1]	27 [4, 3, 4]
9 [3, 4]	28 [1, 4, 1]
10 [4, 2]	29 [2, 1, 2]
11 [4, 1]	30 [1, 2, 1]
12 [1, 2]	31 [2, 4, 2]
13 [3, 2]	32 [2, 3, 4]
14 [2, 1, 4]	33 [2, 4, 1]
15 [1, 3]	34 [4, 1, 2]
16 [3, 1]	35 [3, 4, 1]
17 [4, 3]	36 [3, 1, 2]
18 [4, 2, 4]	37 [4, 3, 2]
19 [4, 1, 4]	38 [4, 2, 1, 4]

39 [4, 1, 3]

40 [2, 3, 1]

41 [2, 4, 3]

42 [4, 1, 4, 3]

43 [3, 2, 1]

44 [3, 4, 2]

45 [1, 3, 1]

46 [2, 3, 2]

47 [1, 4, 1, 4]

48 [3, 4, 3]

49 [3, 2, 3]

50 [2, 1, 2, 4]

51 [1, 2, 1, 4]

52 [3, 1, 3]

53 [2, 4, 2, 4]

54 [4, 3, 1]

55 [1, 3, 2]

56 [4, 2, 4, 3]

57 [1, 4, 3]

58 [4, 2, 3]

59 [4, 1, 2, 4]

60 [4, 3, 4, 2]

61 [1, 2, 3]

62 [3, 1, 2, 4]

63 [4, 3, 2, 4]

64 [1, 4, 2, 1]

65 [4, 1, 3, 4]

66 [2, 1, 4, 2]

67 [2, 3, 1, 4]

68 [4, 2, 4, 1]

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71 [3, 2, 1, 4]

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73 [4, 3, 4, 1]

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75 [2, 3, 2, 4]

76 [2, 1, 2, 1]

77 [3, 4, 3, 4]

78 [4, 2, 4, 2]

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80 [1, 4, 1, 2]

81 [2, 4, 2, 1]

82 [3, 1, 3, 4]

83 [4, 1, 4, 2]
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406 [1, 3, 1, 2, 3, 4, 2]

407 [1, 4, 1, 3, 4, 3, 4]

408 [4, 1, 4, 1, 4, 1, 4,1]

where,

$t_0 =$ (1, 2)(3, 5)(4, 7)(6, 9)(8, 14)(10, 18)(11, 19)(12, 21)(13, 22)(15, 24)(16, 26)(17, 27)(20, 32)(23, 38)(25, 42)(28, 47)(29, 50)(30, 51)(31, 53)(33, 56)(34, 59)(35, 60)(36, 62)(37, 63)(39, 65)(40, 67)(41, 68)(43, 71)(44, 73)(45, 74)(46, 75)(48, 77)(49, 79)(52, 82)(54, 86)(55, 87)(57, 83)(58, 90)(61, 94)(64, 100)(66, 104)(69, 107)(70, 110)(72, 112)(76, 116)(78, 120)(80, 121)(81, 123)(84, 125)(85, 127)(88, 132)(89, 133)(91, 136)(92, 139)(93, 140)(95, 143)(96, 146)(97, 147)(98, 149)(99, 150)(101, 152)(102, 154)(103, 155)(105, 158)(106, 160)(108, 162)(109, 164)(111, 167)(113, 171)(114, 172)(115, 173)(117, 175)(118, 177)(119, 178)(122, 182)(124, 185)(126, 188)(128, 191)(129, 183)(130, 193)(131, 195)(134, 151)(135, 200)(137, 197)(138, 153)(141, 205)(142, 206)(144, 176)(145, 209)(148, 212)(156, 217)(157, 220)(159, 221)(161, 225)(163, 229)(165, 230)(166, 231)(168, 215)(169, 233)(170, 189)(174, 239)(179, 244)(180, 247)(181, 248)(184, 251)(186, 254)(187, 256)(190, 259)(192, 211)(194, 262)(196, 264)(198, 265)(199, 268)(201, 270)(202, 271)(203, 272)(204, 274)(207, 277)(208, 278)(210, 279)(213, 280)(214, 283)(216, 284)(218, 285)(219, 287)(222, 289)(223, 290)(224, 291)(226, 292)(227, 294)(228, 295)(232, 300)(234, 301)(235, 302)(236, 303)(237, 305)(238, 306)(240, 308)(241, 309)(242, 310)(243, 311)(245, 313)(246, 315)(249, 320)(250, 321)(252, 324)(253, 325)(255, 297)(257, 330)(258, 331)(260, 333)(261, 334)(263, 293)(266, 337)(267, 339)(269, 340)(273, 344)(275, 316)(276, 335)(281, 327)(282, 299)(286, 349)(288, 353)(296, 354)(298, 355)(304, 358)(307, 361)(312, 363)(314, 367)(317, 368)(318, 369)(319, 370)(322, 373)(323, 376)(326, 377)(328, 378)(329, 380)(332, 381)(336, 382)(338, 383)(341, 384)(342, 385)(343, 386)(345, 387)(346, 388)(347, 389)(348, 390)(350, 392)(351, 393)(352, 394)(356, 397)(357, 399)(359, 400)(360, 401)(362, 402)(364, 391)(365, 396)(366, 403)(371, 405)(372, 406)(374, 407)(375, 398)(379, 395)(404, 408)

$t_1 =$ (1, 3)(2, 11)(4, 8)(5, 28)(6, 16)(7, 33)(9, 35)(10, 23)(12, 30)(13, 43)(14, 69)(15, 45)(17, 54)(18, 68)(19, 84)(20, 40)(21, 91)(22, 95)(24, 97)(25, 64)(26, 114)(27, 73)(29, 76)(31, 81)(32, 130)(34, 88)(36, 74)(37, 105)(38, 156)(39, 80)(41, 141)(42, 155)(44, 111)(46, 93)(47, 179)(48, 109)(49, 124)(50, 181)(51, 72)(52, 118)(53, 189)(55, 128)(56, 198)(57, 134)(58, 169)(59, 152)(60, 203)(61, 102)(62, 210)(63,213)(65, 139)(66, 151)(67, 223)(70, 161)(71, 234)(75, 236)(77, 160)(78, 174)(79, 227)(82, 249)(83, 183)(85, 186)(86, 184)(87, 263)(89, 196)(90, 269)(92, 201)(94, 222)(96, 207)(98, 121)(99, 215)(100, 137)(101, 165)(103, 232)(104, 283)(106, 206)(107, 296)(108, 219)(110, 298)(112, 282)(113, 212)(115, 241)(116, 159)(117, 242)(119, 257)(120, 316)(122, 252)(123, 322)(125, 328)(126, 240)(127, 332)(129, 260)(131, 177)(132, 192)(133, 289)(135, 153)(136, 342)(138, 146)(140, 344)(142, 275)(143, 331)(144, 200)(145, 221)(147, 318)(148, 150)(149, 288)(154, 295)(157, 277)(158, 324)(162, 293)(163, 291)(164, 307)(166, 195)(167, 356)(168, 190)(170, 262)(171, 271)(172, 359)(173, 308)(175, 360)(176, 285)(178, 284)(180, 312)(182, 371)(185, 325)(187, 326)(188, 309)(191, 305)(193, 278)(194, 216)(197, 335)(199, 315)(202, 341)(204, 280)(205, 336)(208, 286)(209, 310)(211, 346)(214, 343)(217, 383)(218, 350)(220, 392)(224, 294)(225, 233)(226, 311)(228, 345)(229, 246)(230, 281)(231, 254)(235, 290)(237, 357)(238, 301)(239, 361)(243, 337)(244, 404)(245, 338)(247, 369)(248, 401)(250, 259)(251, 377)(253, 268)(255, 365)(256, 397)(258, 375)(261, 270)(264, 390)(265, 391)(266, 339)(267, 303)(272, 380)(273, 292)(274, 370)(276, 317)(279, 363)(287, 376)(297, 385)(299, 396)(300, 406)(302, 395)(304, 340)(306, 402)(313, 382)(314, 400)(319, 405)(320, 381)(321, 384)(323, 399)(327, 393)(329, 367)(330, 373)(333, 394)(334, 389)(347, 353)(348, 387)(349, 379)(351, 388)(352, 386)(354, 366)(355, 374)(358, 407)(362, 398)(364, 403)(368, 372)(378, 408)

$t_2 =$ (1, 4)(2, 10)(3, 12)(5, 25)(6, 13)(7, 31)(8, 29)(9, 44)(11, 34)(14, 66)(15, 55)(16, 36)(17, 37)(18, 78)(19, 83)(20, 46)(21, 89)(22, 93)(23, 70)(24, 106)(26, 113)(27, 60)(28, 80)(30, 85)(32, 129)(33, 92)(35, 96)(38, 153)(39, 142)(40, 98)(41, 99)(42, 163)(43, 75)(45, 114)(47, 176)(48, 122)(49, 115)(50, 61)(51, 184)(52, 108)(53, 187)(54, 131)(56, 197)(57, 166)(58, 81)(59, 202)(62, 208)(63, 181)(64, 157)(65, 216)(67, 222)(68, 147)(69, 165)(71, 232)(72, 135)(73, 237)(74, 204)(76, 180)(77, 193)(79, 218)(82, 253)(84, 190)(86, 261)(87, 241)(88, 199)(90, 100)(91, 150)(94, 276)(95, 211)(97, 214)(101, 167)(102, 152)(103, 139)(104, 286)(105, 173)(107, 293)(109, 224)(110, 148)(111, 235)(112, 263)(116, 307)(117, 258)(118, 223)(119, 238)(120, 314)(121, 319)(123, 128)(124, 250)(125, 327)(126, 243)(127, 194)(130, 233)(132, 336)(133, 338)(134, 209)(136, 333)(137, 210)(138, 196)(140, 343)(141, 158)(143, 217)(144, 221)(145, 174)(146, 347)(149, 272)(151, 192)(154, 170)(155, 299)(156, 288)(159, 269)(160, 301)(161, 297)(162, 294)(164, 351)(168, 262)(169, 281)(171, 308)(172, 358)(175, 290)(177, 331)(178, 340)(179, 317)(182, 332)(183, 323)(185, 289)(186, 329)(188, 374)(189, 320)(191, 341)(195, 291)(198, 324)(200, 337)(201, 302)(203, 345)(205, 247)(206, 268)(207, 309)(212, 352)(213, 220)(215, 311)(219, 239)(225, 227)(226, 274)(228, 292)(229, 395)(230, 398)(231, 298)(234, 342)(236, 334)(240, 348)(242, 295)(244, 389)(245, 296)(246, 362)(248, 368)(249, 372)(251, 407)(252, 325)(254, 376)(255, 265)(256, 404)(257, 278)(259, 381)(260, 359)(264, 275)(266, 357)(267, 310)(270, 406)(271, 354)(273, 284)(277, 391)(279, 392)(280, 397)(282, 326)(283, 388)(285, 361)(287, 353)(300, 380)(303, 373)(304, 339)(305, 363)(306, 370)(312, 378)(313, 355)(315, 349)(316, 405)(318, 387)(321, 369)(322, 393)(328, 386)(330, 396)(335, 402)(344, 401)(346, 400)(350, 377)(356, 384)(360, 382)(364, 385)(365, 390)(366, 383)(367, 408)(371, 394)(375, 399)(379, 403)

$t_3 =$ (1, 6)(2, 17)(3, 15)(4, 20)(5, 57)(7, 41)(8, 72)(9, 48)(10, 58)(11, 39)(12, 61)(13, 49)(14, 144)(16, 52)(18, 56)(19, 42)(21, 170)(22, 103)(23, 159)(24, 108)(25, 145)(26, 137)(27, 119)(28, 88)(29, 89)(30, 69)(31, 70)(32, 124)(33, 168)(34, 194)(35, 101)(36, 148)(37, 109)(38, 281)(40, 79)(43, 192)(44, 138)(45, 117)(46, 126)(47, 171)(50, 273)(51, 304)(53, 143)(54, 122)(55, 82)(59, 275)(60, 104)(62, 222)(63, 151)(64, 231)(65, 218)(66, 282)(67, 260)(68, 228)(71, 263)(73, 136)(74, 156)(75, 202)(76, 234)(77, 245)(78, 235)(80, 226)(81, 267)(83, 266)(84, 207)(85, 208)(86, 150)(87, 214)(90, 249)(91, 276)(92, 205)(93, 157)(94, 240)(95, 158)(96, 206)(97, 153)(98, 100)(99, 219)(102, 164)(105, 261)(106, 191)(107, 274)(110, 360)(111, 233)(112, 258)(113, 195)(114, 201)(115, 246)(116, 278)(118, 255)(120, 280)(121, 356)(123, 347)(125, 334)(127, 301)(128, 139)(129, 152)(130, 149)(131, 213)(132, 374)(133, 303)(134, 250)(135, 182)(140, 293)(141, 238)(142, 175)(146, 315)(147, 300)(154, 350)(155, 348)(160, 248)(161, 344)(162, 326)(163, 345)(165, 306)(166, 257)(167, 297)(169, 188)(172, 289)(173, 319)(174, 331)(176, 309)(177, 322)(178, 366)(179, 343)(180, 272)(181, 352)(183, 279)(184, 346)(185, 317)(186, 305)(187, 359)(189, 290)(190, 308)(193, 251)(196, 330)(197, 375)(198, 357)(199, 358)(200, 372)(203, 225)(204, 291)(209, 371)(210, 338)(211, 265)(212, 229)(215, 351)(216, 287)(217, 370)(220, 385)(221, 339)(223, 318)(224, 254)(227, 327)(230, 394)(232, 296)(236, 324)(237, 268)(239, 363)(241, 323)(242, 364)(243, 379)(244, 393)(247, 252)(253, 316)(256, 405)(259, 380)(262, 292)(264, 388)(269, 321)(270, 349)(271, 373)(277, 401)(283, 395)(284, 355)(285, 400)(286, 387)(288, 354)(294, 361)(295, 396)(298, 406)(299, 384)(302, 407)(307, 389)(310, 390)(311, 398)(312, 376)(313, 404)(314, 377)(320, 386)(325, 381)(328, 368)(329, 369)(332, 397)(333, 391)(335, 353)(336, 392)(337, 402)(340, 382)(341, 383)(342, 399)(362, 367)(365, 378)(403, 408)

Proof of Isomorphism

We obtained the collapsed Cayley graph from the information contained in the following computations

$$\frac{|N|}{|N|}+\frac{|N|}{|N^{(0)}|}+\frac{|N|}{|N^{(01)}|}+\frac{|N|}{|N^{(010)}|}+\frac{|N|}{|N^{(012)}|}+\frac{|N|}{|N^{(013)}|}+\frac{|N|}{|N^{(0101)}|}+\frac{|N|}{|N^{(0102)}|}+\frac{|N|}{|N^{(0103)}|}+\frac{|N|}{|N^{(0120)}|}$$
$$+\frac{|N|}{|N^{(0123)}|}+\frac{|N|}{|N^{(0130)}|}+\frac{|N|}{|N^{(0132)}|}+\frac{|N|}{|N^{(01010)}|}+\frac{|N|}{|N^{(01013)}|}+\frac{|N|}{|N^{(01021)}|}+\frac{|N|}{|N^{(01023)}|}+\frac{|N|}{|N^{(01031)}|}$$
$$+\frac{|N|}{|N^{(01032)}|}+\frac{|N|}{|N^{(01201)}|}+\frac{|N|}{|N^{(01203)}|}+\frac{|N|}{|N^{(01230)}|}+\frac{|N|}{|N^{(01231)}|}+\frac{|N|}{|N^{(01232)}|}+\frac{|N|}{|N^{(01320)}|}+\frac{|N|}{|N^{(010101)}|}$$
$$+\frac{|N|}{|N^{(010102)}|}+\frac{|N|}{|N^{(010103)}|}+\frac{|N|}{|N^{(010130)}|}+\frac{|N|}{|N^{(010132)}|}+\frac{|N|}{|N^{(010212)}|}+\frac{|N|}{|N^{(010213)}|}+\frac{|N|}{|N^{(010231)}|}$$
$$+\frac{|N|}{|N^{(010232)}|}+\frac{|N|}{|N^{(010312)}|}+\frac{|N|}{|N^{(010321)}|}+\frac{|N|}{|N^{(010323)}|}+\frac{|N|}{|N^{(0101010)}|}+\frac{|N|}{|N^{(0101012)}|}+\frac{|N|}{|N^{(0101013)}|}$$
$$+\frac{|N|}{|N^{(0101303)}|}+\frac{|N|}{|N^{(0102121)}|}+\frac{|N|}{|N^{(0102132)}|}+\frac{|N|}{|N^{(01010101)}|}$$

$$=\frac{12}{12}+\frac{12}{3}+\frac{12}{1}+\frac{12}{1}+\frac{12}{1}+\frac{12}{1}+\frac{12}{1}+\frac{12}{1}+\frac{12}{1}+\frac{12}{1}+\frac{12}{1}+\frac{12}{1}+\frac{12}{1}$$
$$+\frac{12}{1}+\frac{12}{1}+\frac{12}{1}+\frac{12}{1}+\frac{12}{1}+\frac{12}{1}+\frac{12}{2}+\frac{12}{1}+\frac{12}{3}+\frac{12}{1}+\frac{12}{1}+\frac{12}{3}+\frac{12}{1}$$
$$+\frac{12}{1}+\frac{12}{1}+\frac{12}{1}+\frac{12}{2}+\frac{12}{1}+\frac{12}{1}+\frac{12}{3}+\frac{12}{3}+\frac{12}{1}+\frac{12}{3}+\frac{12}{3}+\frac{12}{3}+\frac{12}{1}$$
$$+\frac{12}{1}+\frac{12}{3}+\frac{12}{3}+\frac{12}{2}+\frac{12}{12}$$

$$= 1 + 4 + 12 + 12 + 12 + 12 + 12 + 12 + 12 + 12 + 12 + 12 + 12 + 12 + 12 + 12$$
$$+ 12 + 12 + 12 + 6 + 12 + 4 + 12 + 12 + 4 + 12 + 12 + 12 + 12 + 6 + 12 + 12 + 4$$
$$+ 4 + 12 + 4 + 4 + 4 + 12 + 12 + 4 + 4 + 6 + 1$$
$$= 408$$

this tells us that the maximum possible index of N in G is 408. It follows that the order of the image group G is at most $|N| * 408 = 12 * 408 = 4{,}896$. The order of G can be confirmed by regarding G as a permutation group on the 408 cosets that we have found.

The action of the control group N on the cosets is

x: (3, 4, 6)(5, 7, 9)(8, 13, 15)(10, 17, 11)(12, 20, 16)(14, 22, 24)(18, 27, 19)(21, 32, 26)(23, 37, 39)(25, 41, 35)(28, 31, 48)(29, 49, 45)(30, 46, 52)(33, 44, 57)(34, 58, 54)(36, 61, 40)(38, 63, 65)(42, 68, 60)(43, 55, 72)(47, 53, 77)(50, 79, 74)(51, 75, 82)(56, 73, 83)(59, 90, 86)(62, 94, 67)(64, 99, 101)(66, 103, 97)(69, 93, 108)(70, 109, 80)(71, 87, 112)(76, 115, 117)(78, 119, 84)(81, 122, 88)(85, 126, 118)(89, 124, 114)(91, 129, 137)(92, 138, 134)(95, 106, 144)(96, 145, 141)(98, 148, 102)(100, 150, 152)(104, 155, 147)(105, 142, 159)(107, 140, 162)(110, 164, 121)(111, 166, 168)(113, 170, 130)(116, 173, 175)(120, 178, 125)(123, 182, 132)(127, 188, 177)(128, 135, 192)(131, 194, 169)(133, 185, 172)(136, 183, 197)(139, 153, 151)(143, 160, 176)(146, 209, 205)(149, 212, 154)(156, 181, 218)(157, 219, 165)(158, 206, 221)(161, 224, 226)(163, 228, 203)(167, 231, 215)(171, 189, 193)(174, 238, 207)(179, 187, 245)(180, 246, 242)(184, 202, 249)(186, 243, 255)(190, 235, 257)(191, 200, 211)(195, 262, 233)(196, 250, 201)(198, 237, 266)(199, 267, 252)(204, 273, 227)(208, 240, 223)(210, 276, 260)(213, 216, 281)(214, 282, 232)(217, 248, 285)(220, 287, 230)(225, 291, 292)(229, 295, 272)(234, 241, 258)(236, 253, 304)(239, 306, 277)(244, 256, 313)(247, 315, 310)(251, 271, 320)(254, 311, 297)(259, 302, 330)(261, 275, 269)(264, 321, 270)(265, 305, 337)(268, 339, 324)(274, 344, 294)(278, 308, 290)(279, 335, 333)(280, 284, 327)(283, 299, 300)(286, 348, 318)(288, 352, 350)(296, 343, 326)(298, 351, 356)(301, 309, 331)(303, 325, 358)(307, 319, 360)(312, 362, 364)(314, 366, 328)(316, 340, 334)(317, 359, 338)(322, 332, 374)(323, 375, 342)(329, 379, 365)(336, 347, 371)(341, 372, 346)(349, 390, 369)(353, 394, 392)(354, 386, 377)(355, 393, 397)(361, 370, 401)(363, 402, 391)(367, 403, 378)(368, 400, 383)(373, 381, 407)(376, 398, 385)(380, 395, 396)(382, 389, 405)(384, 406, 388)

y: (2, 3, 4)(5, 8, 10)(7, 11, 12)(9, 16, 13)(14, 23,25)(15, 20, 17)(18, 28, 29)(19, 30, 31)(21, 33, 34)(22, 35, 36)(24, 40, 37)(26, 43, 44)(27, 45, 46)(32, 54, 55)(38, 64, 66)(39, 61, 41)(42, 69, 70)(47, 76, 78)(48, 52, 49)(50, 68, 80)(51, 81, 83)(53, 84, 85)(56, 88, 89)(57, 72, 58)(59, 91, 92)(60, 74, 93)(62, 95, 96)(63, 97, 98)(65, 102, 99)(67, 105, 106)(71, 111, 113)(73, 114, 75)(77, 118, 115)(79, 109, 108)(82, 124, 122)(86, 128, 129)(87, 130, 131)(90, 134, 135)(94, 141, 142)(100, 151, 153)(101, 148, 103)(104, 156, 157)(107, 161, 163)(110, 155, 165)(112, 169, 166)(116, 174, 176)(117, 126, 119)(120, 179, 180)(121, 181, 147)(123, 183, 184)(125, 186, 187)(127, 189, 190)(132, 196, 197)(133, 198, 199)(136, 201, 202)(137, 192, 138)(139, 152, 150)(140, 203, 204)(143, 207, 208)(144, 159, 145)(146, 210, 211)(149, 213, 214)(154, 215, 216)(158, 206, 222)(160, 223, 173)(162, 227, 224)(164, 219, 218)(167, 212, 232)(168, 194, 170)(171, 234, 235)(172, 236, 237)(175, 240, 238)(177, 241, 193)(178, 242, 243)(182, 249, 250)(185, 252, 253)(188, 257, 258)(191, 260, 261)(195, 263, 233)(200, 269, 209)(205, 275, 276)(217, 277, 286)(220, 283, 288)(225, 291, 293)(226, 273, 228)(229, 296, 297)(230, 298, 299)(231, 282, 281)(239, 285, 307)(244, 312, 314)(245, 255, 246)(247, 316, 317)(248, 318, 319)(251, 322, 323)(254, 326, 327)(256, 328, 329)(259, 332, 320)(264, 335, 336)(265, 315, 338)(266, 304, 267)(268, 289, 324)(270, 341, 333)(271, 342, 302)(272, 280, 343)(274, 344, 345)(278, 331, 309)(279, 346, 347)(284, 295, 311)(287, 350, 351)(290, 308, 301)(300, 356, 352)(303, 357, 358)(305, 359, 334)(306, 360, 348)(310, 337, 340)(313, 365, 362)(321, 371, 372)(330, 375, 374)(349, 383, 391)(353, 392, 388)(354, 385, 395)(355, 396, 398)(363, 400, 389)(364, 379, 366)(367, 404, 378)(368, 369, 405)(370, 401, 387)(373, 399, 407)(376, 377, 393)(380, 397, 386)(382, 390, 402)(384, 394, 406)

The action of x and y, and hence N, on these symmetric generators is as follows:

x: (t_1, t_2, t_3)

y: (t_0, t_1, t_2)

The action of the symmetric generator t_0 on the cosets is:

$t_0 =$ (1, 2)(3, 5)(4, 7)(6, 9)(8, 14)(10, 18)(11, 19)(12, 21)(13, 22)(15, 24)(16, 26)(17, 27)(20, 32)(23, 38)(25, 42)(28, 47)(29, 50)(30, 51)(31, 53)(33, 56)(34, 59)(35, 60)(36, 62)(37, 63)(39, 65)(40, 67)(41, 68)(43, 71)(44, 73)(45, 74)(46, 75)(48, 77)(49, 79)(52, 82)(54, 86)(55, 87)(57, 83)(58, 90)(61, 94)(64, 100)(66, 104)(69, 107)(70, 110)(72, 112)(76, 116)(78, 120)(80, 121)(81, 123)(84, 125)(85, 127)(88, 132)(89, 133)(91, 136)(92, 139)(93, 140)(95, 143)(96, 146)(97, 147)(98, 149)(99, 150)(101, 152)(102, 154)(103, 155)(105, 158)(106, 160)(108, 162)(109, 164)(111, 167)(113, 171)(114, 172)(115, 173)(117, 175)(118, 177)(119, 178)(122, 182)(124, 185)(126, 188)(128, 191)(129, 183)(130, 193)(131, 195)(134, 151)(135, 200)(137, 197)(138, 153)(141, 205)(142, 206)(144, 176)(145, 209)(148, 212)(156, 217)(157, 220)(159, 221)(161, 225)(163, 229)(165, 230)(166, 231)(168, 215)(169, 233)(170, 189)(174, 239)(179, 244)(180, 247)(181, 248)(184, 251)(186, 254)(187, 256)(190, 259)(192, 211)(194, 262)(196, 264)(198, 265)(199, 268)(201, 270)(202, 271)(203, 272)(204, 274)(207, 277)(208, 278)(210, 279)(213, 280)(214, 283)(216, 284)(218, 285)(219, 287)(222, 289)(223, 290)(224, 291)(226, 292)(227, 294)(228, 295)(232, 300)(234, 301)(235, 302)(236, 303)(237, 305)(238, 306)(240, 308)(241, 309)(242, 310)(243, 311)(245, 313)(246, 315)(249, 320)(250, 321)(252, 324)(253, 325)(255, 297)(257, 330)(258, 331)(260, 333)(261, 334)(263, 293)(266, 337)(267, 339)(269, 340)(273, 344)(275, 316)(276, 335)(281, 327)(282, 299)(286, 349)(288, 353)(296, 354)(298, 355)(304, 358)(307, 361)(312, 363)(314, 367)(317, 368)(318, 369)(319, 370)(322, 373)(323, 376)(326, 377)(328, 378)(329, 380)(332, 381)(336, 382)(338, 383)(341, 384)(342, 385)(343, 386)(345, 387)(346, 388)(347, 389)(348, 390)(350, 392)(351, 393)(352, 394)(356, 397)(357, 399)(359, 400)(360, 401)(362, 402)(364, 391)(365, 396)(366, 403)(371, 405)(372, 406)(374, 407)(375, 398)(379, 395)(404, 408)

We note that xy has order 2, and hence $N=\langle x,y\rangle \cong A_4$.

Now we check our relations; that is

$t_0^{t_2 t_1 t_0 t_2 t_1 t_0 t_2 t_1 t_0 t_2 t_1 t_0 t_2 t_1 t_0 t_2 t_1 t_0} = t_0$

$t_1^{t_2 t_1 t_0 t_2 t_1 t_0 t_2 t_1 t_0 t_2 t_1 t_0 t_2 t_1 t_0 t_2 t_1 t_0} = t_1$

$t_2^{t_2 t_1 t_0 t_2 t_1 t_0 t_2 t_1 t_0 t_2 t_1 t_0 t_2 t_1 t_0 t_2 t_1 t_0} = t_2$

$t_3^{t_2 t_1 t_0 t_2 t_1 t_0 t_2 t_1 t_0 t_2 t_1 t_0 t_2 t_1 t_0 t_2 t_1 t_0} = t_3$

This means that $t_2\, t_1\, t_0\, t_2\, t_1\, t_0\, t_2\, t_1\, t_0\, t_2\, t_1\, t_0\, t_2\, t_1\, t_0\, t_2\, t_1\, t_0$ fixes all the symmetric generators, so it acts as the identity on t_0, t_1, t_2, t_3, that is $t_2\, t_1\, t_0\, t_2\, t_1\, t_0\, t_2\, t_1\, t_0\, t_2\, t_1\, t_0\, t_2\, t_1\, t_0\, t_2\, t_1\, t_0 = 1$ or equivalently $t_2\, t_1\, t_0\, t_2\, t_1\, t_0\, t_2\, t_1\, t_0 = t_0\, t_1\, t_2\, t_0\, t_1\, t_2\, t_0\, t_1\, t_2$, this proves that relation (1) holds in $PGL_2(17)$.

Similarly,

$t_0^{t_1 t_0 t_1 t_0 t_1 t_0 t_1 t_0 t_1 t_0 t_1 t_0 t_1 t_0 t_1 t_0} = t_0$

$t_1^{t_1 t_0 t_1 t_0 t_1 t_0 t_1 t_0 t_1 t_0 t_1 t_0 t_1 t_0 t_1 t_0} = t_1$

$t_2^{t_1 t_0 t_1 t_0 t_1 t_0 t_1 t_0 t_1 t_0 t_1 t_0 t_1 t_0 t_1 t_0} = t_2$

$t_3^{t_1 t_0 t_1 t_0 t_1 t_0 t_1 t_0 t_1 t_0 t_1 t_0 t_1 t_0 t_1 t_0} = t_3$

This means that $t_1\, t_0\, t_1\, t_0\, t_1\, t_0\, t_1\, t_0\, t_1\, t_0\, t_1\, t_0\, t_1\, t_0\, t_1\, t_0$ fixes all the symmetric generators, so it acts as the identity on t_0, t_1, t_2, t_3, that is $t_1\, t_0\, t_1\, t_0\, t_1\, t_0\, t_1\, t_0\, t_1\, t_0\, t_1\, t_0\, t_1\, t_0\, t_1\, t_0 = 1$ or equivalently $t_1 t_0 t_1 t_0 t_1 t_0 t_1 t_0 = t_0 t_1 t_0 t_1 t_0 t_1 t_0 t_1$, and that proves our second relation.

Also,

$t_0^{t_3 t_0 t_3 t_1 t_3 t_2 t_3 t_0} = t_1$

$t_1^{t_3 t_0 t_3 t_1 t_3 t_2 t_3 t_0} = t_2$

$t_2^{t_3 t_0 t_3 t_1 t_3 t_2 t_3 t_0} = t_0$

$t_3^{t_3 t_0 t_3 t_1 t_3 t_2 t_3 t_0} = t_3$

Thus $t_3 t_0 t_3 t_1 t_3 t_2 t_3 t_0$ acts as the permutation (0 1 2) on the symmetric generators, that is $t_3 t_0 t_3 t_1 t_3 t_2 t_3 t_0 = (0\ 1\ 2)$, hence relation (3) holds in $PGL_2(17)$.

We note that the elements x, y and t_0 generate the whole group, $PGL_2(17)$, and so $PGL_2(17)$ is an image of G. Thus, $|G| \geq |PGL_2(17)|$.

But we also have that $|G| \leq 4{,}896 = |PGL_2(17)|$.

Therefore, $|G| \leq 4{,}896 = |PGL_2(17)| \leq |G|$,

which proves the isomorphism, that is $G \cong PGL_2(17)$.

APPENDIX

MAGMA WORK

Magma work for the group A_5 over S_3

```
S3:=Sym(3);
x:=S3!(1,2,3);
y:=S3!(1,2);
G<x,y,t>:=Group<x,y,t|x^3,y^2,(x*y)^2,t^2, (t,y),(x*t)^5 ,
(y^(x^-1)*t)^5 >;
print Order(G);
60
print Index(G, sub<G|x,y>:CosetLimit:=5000000,Hard:=true);
10
S10:=Sym(10);
a:=S10!(1, 2)(3, 6)(4, 8)(9, 10);
b:=S10!(1, 3)(2, 5)(4, 10)(8, 7);
c:=S10!(1, 4)(2, 7)(3, 9)(5, 6);
N:=sub<S10|a,b,c>;
print Order(N);
60
```

Magma work for the group $PGL_2(7)$ over S_4

```
S4:=Sym(4);
x:=S4!(4,1,2,3);
y:=S4!(2,3);
x:=S4!(4,1,2,3);
y:=S4!(2,3);
G:=sub<S4|x,y>;
F<x,y>:=FPGroup(G);
F;
Finitely presented group F on 2 generators
Relations
    x^4 = Id(F)
    y^2 = Id(F)
    (y * x)^3 = Id(F)
N4:=Stabiliser(S4,4);
N41:=Stabiliser(N4,1);
N412:=Stabiliser(N41,2);
Cent:=Centraliser(S4,N412);
for i in Cent do print i; end for;
Id(S4)
(1, 2, 3, 4)
(1, 4, 3, 2)
(1, 3)(2, 4)
(2, 3)
(1, 2, 4)
(1, 4, 3)
```

```
(1, 3, 4, 2)
(2, 4, 3)
(1, 2)
(1, 4, 2, 3)
(1, 3, 4)
(3, 4)
(1, 2, 3)
(1, 4, 2)
(1, 3, 2, 4)
(2, 3, 4)
(1, 2, 4, 3)
(1, 4)
(1, 3, 2)
(2, 4)
(1, 2)(3, 4)
(1, 4)(2, 3)
(1, 3)
```

Magma work for the group G over S_4

```
S16:=Sym(16);
a:=S16!(1, 2)(3, 6)(4, 7)(5, 8)(9, 12)(10, 13)(11, 14)(15,
16);
b:=S16!(1, 3)(2, 6)(4, 9)(5, 10)(7, 12)(8, 13)(11, 15)(14,
16);
c:=S16!(1, 4)(3, 9)(5, 11)(6, 12)(7, 2)(8, 14)(10, 15)(13,
16);
d:=S16!(1, 5)(3, 10)(4, 11)(6, 13)(7, 14)(8, 2)(9, 15)(12,
16);
N:=sub<S16|a,b,c,d>;
print Order(N);
16
```

Magma work for the group G over S_6

```
S14:=Sym(14);
a:=S14!(1, 2)(3, 9)(4, 10)(5, 11)(6, 12)(7, 13)(8, 14);
b:=S14!(1, 3)(2, 8)(4, 10)(5, 11)(6, 12)(7, 13)(9, 14);
c:=S14!(1, 4)(2, 8)(3, 9)(5, 11)(6, 12)(7 ,13)(10 ,14);
d:=S14!(1 ,5)(2 ,8)(3, 9)(4 ,10)(6 ,12)(7 ,13)(11, 14);
e:=S14!(1, 6)(2 ,8)(3, 9)(4 ,10)(5 ,11)(7 ,13)(12, 14);
f:=S14!(1 ,7)(2 ,8)(3 ,9)(4 ,10)(5 ,11)(6 ,12)(13, 14);
N:=sub<S14|a,b,c,d,e,f>;
Print Order (G);
```

Magma work for the Group $PGL_2(11)$ over $L_2(5)$

```
S12:=Sym(12);
a:=S12!(1,2,3,4,5,6,7,8,9,10,11);
b:=S12!(1,3,9,5,4)(2,6,7,10,8);
c:=S12!(11,12)(1,10)(2,5)(3,7)(4,8)(6,9);
L211:=sub<S12|a,b,c>;
d:=S12!(1,12)(2,10)(3,11)(5,6)(7,8);
PGL211:=sub<S12|L211,d>;
e:=PGL211!(1, 3)(2, 12)(4, 7)(5, 9)(6, 10)(8, 11);
f:=PGL211!(1, 3, 8)(2, 10, 12)(4, 9, 11)(5, 6, 7);
L:=sub<PGL211|e,f>;
print f*e;
(2, 6, 4, 5, 10)(3, 11, 7, 9, 8)
t:=PGL211!(2, 11)(3, 10)(4, 9)(5, 8)(6, 7);
for i in [1..5] do print (t)^(f*e)^i; end for;
(2, 11)(3, 10)(4, 9)(5, 8)(6, 7)
(2, 11)(3, 10)(4, 9)(5, 8)(6, 7)
(2, 11)(3, 10)(4, 9)(5, 8)(6, 7)
(2, 11)(3, 10)(4, 9)(5, 8)(6, 7)
(2, 11)(3, 10)(4, 9)(5, 8)(6, 7)
t:=PGL211!(1, 5)(2, 8)(3, 9)(6, 10)(11, 12);
for i in [1..5] do print (t)^(f*e)^i; end for;
(1, 10)(2, 4)(3, 6)(7, 12)(8, 11)
(1, 2)(3, 7)(4, 11)(5, 6)(9, 12)
(1, 6)(4, 10)(5, 7)(8, 12)(9, 11)
(1, 4)(2, 5)(3, 12)(7, 8)(9, 10)
(1, 5)(2, 8)(3, 9)(6, 10)(11, 12)
C:=Class(PGL211, PGL211!(1,3,8)(2,10,12)(4,9,11)(5,6,7));
for g in C do if Order((f*e)*g) eq 2 then print g; end if;
end for;
(1, 7, 9)(2, 8, 11)(3, 10, 6)(4, 5, 12)
(1, 3, 11)(2, 6, 12)(4, 10, 9)(5, 7, 8)
(1, 5, 10)(2, 4, 9)(3, 11, 12)(6, 7, 8)
(1, 9, 8)(2, 4, 11)(3, 7, 6)(5, 10, 12)
(1, 11, 7)(2, 8, 5)(3, 10, 9)(4, 12, 6)
(1, 8, 3)(2, 12, 10)(4, 11, 9)(5, 7, 6)
(1, 2, 6)(3, 4, 10)(5, 8, 11)(7, 9, 12)
(1, 10, 2)(3, 4, 9)(5, 8, 6)(7, 12, 11)
(1, 6, 4)(2, 11, 5)(3, 7, 10)(8, 12, 9)
(1, 4, 5)(2, 11, 9)(3, 12, 8)(6, 7, 10)
xx:=f*e;
yy:=PGL211!(1, 8, 3)(2, 12, 10)(4, 11, 9)(5, 7, 6);
ll:=sub<PGL211|xx,yy>;
print Order(L);
```

```
60
print Order(l1);
60
IsIsomorphic(L,l1);
true Homomorphism of GrpPerm: $, Degree 12 into GrpPerm: $,
Degree 12, Order 2^2
* 3 * 5 induced by
    (1, 5)(2, 6)(3, 11)(4, 7)(8, 10)(9, 12) |--> (1, 8)(2,
5)(3, 9)(4, 7)(6,
        11)(10, 12)
    (1, 3, 11)(2, 6, 12)(4, 10, 9)(5, 7, 8) |--> (1, 5,
2)(3, 12, 9)(4, 6, 8)(7,
        11, 10)
rec<recformat<permgroup: GrpPerm, fpgroup: GrpFP, fptoperm:
Map, autgroup:
GrpFP, outerautgroup: GrpFP, fptoaut: Map, auttoouter: Map,
orderautgroup:
RngIntElt, orderouterautgroup: RngIntElt, centre: GrpFP,
newgroup: GrpPerm,
radquot: GrpPerm, radmap: Map, radinvars: Tup, rqwordgp:
GrpFP, rqgenlist:
SeqEnum, rqprojlist: SeqEnum, rqfplist: SeqEnum, rqsocquot:
GrpPerm, rqsocmap:
Map, rqsqwordmap: Map, subseries: SeqEnum, length:
RngIntElt, radindex:
RngIntElt, index: SeqEnum, layermod: SeqEnum, layermap:
SeqEnum, quotgens:
SeqEnum, split: BoolElt, outimages: SeqEnum, outautos:
List, genims: SeqEnum,
soluble: BoolElt, printlevel: RngIntElt, smallmodaut:
RngIntElt,
verysmallmodaut: RngIntElt, smallouterautgroup: RngIntElt,
smallsuboutgp:
RngIntElt, printsct: RngIntElt, smallsimplefactor:
RngIntElt, oldfpgroup: GrpFP,
oldfptoperm: Map, oldoutimages: SeqEnum, relvals: SeqEnum,
cem: ModMatRngElt,
innerder: List, derspace: List, innermodaut: SeqEnum,
modaut: GrpMat, mapres:
GrpFP, mapresmap: Map, rmamap: Map, liftoutaut: GrpFP,
orderliftoutaut:
RngIntElt, holgens: SeqEnum, holmap: Map, holperm: SeqEnum,
holword: SeqEnum, gpholpt: SeqEnum, newgpholpt: SeqEnum,
imholpt: SeqEnum, newimholpt: SeqEnum> |
```

```
permgroup := Permutation group 11 acting on a set of
cardinality 12
Order = 60 = 2^2 * 3 * 5
    (2, 6, 4, 5, 10)(3, 11, 7, 9, 8)
    (1, 8, 3)(2, 12, 10)(4, 11, 9)(5, 7, 6), fpgroup :=
Finitely presented group
on 2 generators
Relations
    $.1^2 = Id($)
    $.2^3 = Id($)
    ($.2 * $.1)^5 = Id($), fptoperm := Homomorphism of
GrpFP into GrpPerm: $,
Degree 12, Order 2^2 * 3 * 5 induced by
    $.1 |--> (1, 8)(2, 5)(3, 9)(4, 7)(6, 11)(10, 12)
    $.2 |--> (1, 5, 2)(3, 12, 9)(4, 6, 8)(7, 11, 10),
autgroup := Finitely
presented group on 3 generators
Relations
    $.1^2 = Id($)
    $.2^3 = Id($)
    ($.2 * $.1)^5 = Id($)
    $.1^$.3 = $.1 * $.2^-1 * $.1 * $.2 * $.1
    $.2^$.3 = $.2 * $.1 * $.2^-1 * $.1 * $.2^-1
    $.3^2 = $.1 * $.2^-1 * $.1 * $.2 * $.1 * $.2,
outerautgroup := Finitely
presented group on 1 generator
Relations
    $.1^2 = Id($), fptoaut := Homomorphism of GrpFP into
GrpFP induced by
    $.1 |--> $.1
    $.2 |--> $.2, auttoouter := Homomorphism of GrpFP into
GrpFP induced by
    $.1 |--> Id($)
    $.2 |--> Id($)
    $.3 |--> $.1, orderautgroup := 120, orderouterautgroup
:= 2, centre :=
Finitely presented group on 1 generator
Generators as words
    $.1 = Id($), newgroup := Permutation group acting on a
set of cardinality 12
Order = 60 = 2^2 * 3 * 5
    (1, 8)(2, 5)(3, 9)(4, 7)(6, 11)(10, 12)
    (1, 5, 2)(3, 12, 9)(4, 6, 8)(7, 11, 10), radquot :=
Permutation group acting on a set of cardinality 12
```

```
Order = 60 = 2^2 * 3 * 5
    (1, 8)(2, 5)(3, 9)(4, 7)(6, 11)(10, 12)
    (1, 5, 2)(3, 12, 9)(4, 6, 8)(7, 11, 10), radmap :=
Mapping from: GrpPerm: ll
to GrpPerm: ll, rqwordgp := Finitely presented group on 2
generators
Relations
    $.1^2 = Id($)
    $.2^3 = Id($)
    ($.2 * $.1)^5 = Id($), rqgenlist := [
    [
        (1, 8)(2, 5)(3, 9)(4, 7)(6, 11)(10, 12),
        (1, 5, 2)(3, 12, 9)(4, 6, 8)(7, 11, 10)
    ],
    []
], rqprojlist := [
    Homomorphism of GrpPerm: $, Degree 12, Order 2^2 * 3 *
5 into GrpPerm: $,
    Degree 12, Order 2^2 * 3 * 5 induced by
        (1, 12)(2, 3)(4, 9)(5, 7)(6, 8)(10, 11) |--> (1,
12)(2, 3)(4, 9)(5,
            7)(6, 8)(10, 11)
        (2, 6, 4, 5, 10)(3, 11, 7, 9, 8) |--> (2, 6, 4, 5,
10)(3, 11, 7, 9, 8)
        (1, 8, 3)(2, 12, 10)(4, 11, 9)(5, 7, 6) |--> (1, 8,
3)(2, 12, 10)(4, 11,
            9)(5, 7, 6)
], rqfplist := [
    Mapping from: GrpFP to GrpPerm: $, Degree 12, Order 2^2
* 3 * 5
], rqsocquot := Permutation group acting on a set of
cardinality 1
Order = 1, rqsocmap := Mapping from: GrpPerm: ll to
GrpPerm: $, Degree 1, Order
1, rqsqwordmap := Mapping from: GrpFP to GrpPerm: $, Degree
1, Order 1,
subseries := [
    Permutation group acting on a set of cardinality 12
    Order = 1
], length := 0, radindex := 60, index := [], layermod :=
[], layermap := [],
quotgens := [ 2 ], outimages := [
    [ $.1 * $.2^-1 * $.1 * $.2 * $.1, $.2 * $.1 * $.2^-1 *
$.1 * $.2^-1 ]
```

```
], outautos := [*
Endomorphism of GrpPerm: $, Degree 12, Order 2^2 * 3 * 5
induced by
    (1, 8)(2, 5)(3, 9)(4, 7)(6, 11)(10, 12) |--> (1, 12)(2,
7)(3, 4)(5, 8)(6,
        11)(9, 10)
    (1, 5, 2)(3, 12, 9)(4, 6, 8)(7, 11, 10) |--> (1, 5,
6)(2, 7, 10)(3, 4, 8)(9,
        11, 12),
Mapping from: GrpPerm: $, Degree 12, Order 2^2 * 3 * 5 to
GrpPerm: $, Degree 12,
Order 2^2 * 3 * 5
*], genims := [
    [
        (1, 9, 6, 4, 3)(2, 12, 5, 11, 7),
        (1, 9, 8)(2, 4, 11)(3, 7, 6)(5, 10, 12)
    ],
    [
        (1, 8, 6, 2, 7)(3, 4, 12, 10, 11),
        (1, 9, 7)(2, 11, 8)(3, 6, 10)(4, 12, 5)
    ],
    [
        (1, 6, 3, 9, 4)(2, 5, 7, 12, 11),
        (1, 3, 11)(2, 6, 12)(4, 10, 9)(5, 7, 8)
    ]
], soluble := false, printlevel := 0, smallmodaut := 1000,
verysmallmodaut := 1,
smallouterautgroup := 20000, smallsuboutgp := 100000,
printsct := 1000,
smallsimplefactor := 5000>
> tinf:=PGL211!(2, 11)(3, 10)(4, 9)(5, 8)(6, 7);
for i in [1..3] do print (tinf)^(yy^i); end for;
(1, 2)(3, 7)(4, 11)(5, 6)(9, 12)
(1, 6)(4, 10)(5, 7)(8, 12)(9, 11)
(2, 11)(3, 10)(4, 9)(5, 8)(6, 7)
tt:=PGL211!(1, 10)(2, 4)(3, 6)(7, 12)(8, 11);
for i in [1..3] do print (tt)^(yy^i); end for;
(1, 5)(2, 8)(3, 9)(6, 10)(11, 12)
(1, 4)(2, 5)(3, 12)(7, 8)(9, 10)
(1, 10)(2, 4)(3, 6)(7, 12)(8, 11)
tinf:=PGL211!(2, 11)(3, 10)(4, 9)(5, 8)(6, 7);
t0:=PGL211!(1, 2)(3, 7)(4, 11)(5, 6)(9, 12);
t1:=PGL211!(1, 6)(4, 10)(5, 7)(8, 12)(9, 11);
t2:=PGL211!(1, 4)(2, 5)(3, 12)(7, 8)(9, 10);
```

```
t3:=PGL211!(1, 5)(2, 8)(3, 9)(6, 10)(11, 12);
t4:=PGL211!(1, 10)(2, 4)(3, 6)(7, 12)(8, 11);
>for i in [5..9] do print (t0)^(xx^i); end for;
(1, 2)(3, 7)(4, 11)(5, 6)(9, 12)
(1, 6)(4, 10)(5, 7)(8, 12)(9, 11)
(1, 4)(2, 5)(3, 12)(7, 8)(9, 10)
(1, 5)(2, 8)(3, 9)(6, 10)(11, 12)
(1, 10)(2, 4)(3, 6)(7, 12)(8, 11)
>print (xx*(t^yy))^4;
Id(PGL211)
print (((t^(yy^2))^xx)*(((t^(yy^2))^xx)^(yy^2)))^3;
(1, 8)(2, 5)(3, 9)(4, 7)(6, 11)(10, 12)
print xx*yy;
(1, 8)(2, 5)(3, 9)(4, 7)(6, 11)(10, 12)
```

Magma work for the Group $PGL_2(7)$ over S_3

```
S3:=Sym(3);
 x:=S3!(1,2,3);
 y:=S3!(1,2);
G<x,y,t>:=Group<x,y,t|x^3,y^2,(x*y)^2,t^2, (t,y),(x*t)^8 ,
(y^(x^-1)*t)^7 , (y^(x^-1)*t*t^(x^2))^6  ,  (y^(x^-
1)*t*t^(x^2)*t)^4>;
print Order(G);
336
print Index(G, sub<G|x,y>:CosetLimit:=5000000,Hard:=true);
56
S3:=Sym(3);
x:=S3!(1,2,3);
y:=S3!(1,2);
N3:=Stabiliser(S3,3);
N31:=Stabiliser(N3,1);
C:=Centraliser(S3,N31);
print C;
Symmetric group S3 acting on a set of cardinality 3
Order = 6 = 2 * 3
    (1, 2, 3)
    (1, 2)
for i in C do print i; end for;
Id(S3)
(1, 2, 3)
(1, 3, 2)
(2, 3)
(1, 2)
(1, 3)
```

```
f, G1, k := CosetAction(G, sub<G|x,y>);
N:=sub<G1|f(x),f(y)>;
print Order(N);
6
N:=sub<S3|x,y>;
C:=Centralizer(N,y^(x^-1));
T:=Transversal(N,C);
print #T;
3
for i in [1..3] do print (y^(x^-1))^T[i],
[3,1,3,1,3,1,3]^T[i]; end for;
(1, 3)
[ 3, 1, 3, 1, 3, 1, 3 ]
(1, 2)
[ 1, 2, 1, 2, 1, 2, 1 ]
(2, 3)
[ 2, 3, 2, 3, 2, 3, 2 ]
C:=Centralizer(N,x);
T:=Transversal(N,C);
print #T;
2
for i in [1..2] do print y^T[i], [3,1,2,3,1,2,3,1]^T[i];
end for;
(1, 2)
[ 3, 1, 2, 3, 1, 2, 3, 1 ]
(1, 2)
[ 3, 2, 1, 3, 2, 1, 3, 2 ]
f,G1,k:=CosetAction(G, sub<G|x,y>);
print Order(G1);
336
H:=sub<G1|f(t),f(t^x),f(t^(x^2))>;
print Order(H);
336
f, G1, k := CosetAction(G, sub<G|x,y>);
N:=sub<G1|f(t),f(t^x),f(t^(x^2))>;
print Order(N);
  336
print f(t);
(1, 2)(3, 5)(4, 6)(7, 11)(8, 12)(9, 13)(10, 14)(15, 23)(16,
24)(18, 25)(19, 26)(21, 27)(22, 28)(29, 38)(31, 39)(32,
40)(33, 41)(34, 42)(35, 43)(36, 44)(45, 52)(46, 50)(48,
49)(51, 56)(53, 54)
print f(t^x);
```

```
(1, 3)(2, 9)(4, 7)(5, 17)(6, 21)(8, 15)(10, 19)(11, 30)(12,
32)(14, 29)(16, 28)(20, 34)(22, 36)(23, 45)(24, 47)(25,
51)(26, 52)(33, 49)(35, 41)(37, 54)(38, 40)(39, 50)(42,
56)(43, 55)(46, 53)
print f(t^(x^2));
(1, 4)(2, 8)(3, 10)(5, 16)(6, 20)(7, 18)(9, 22)(11, 29)(13,
35)(14, 37)(15, 31)(17, 33)(21, 23)(25, 50)(26, 51)(27,
55)(28, 46)(30, 48)(32, 42)(34, 53)(38, 43)(40, 47)(41,
56)(44, 52)(45, 49)
```

Magma work for the Group $L_2(11)$ over A_4

```
S4:=Sym(4);
x:=S4!(1,2,3);
y:=S4!(4,1,2);
G<x,y,t>:=Group<x,y,t|x^3,y^3,(x*y)^2,t^2, (t,x),(y*t)^11 ,
(x*y*t)^5 , (y*t*(t^(y^2))^x)^5, (x*y*t*t^(y^2))^6>;
print Order(G);
660
print Index(G, sub<G|x,y>:CosetLimit:=5000000,Hard:=true);
55
f, G1, k := CosetAction(G, sub<G|x,y>);
N:=sub<G1|f(x),f(y)>;
for g1 in N do for g2 in N do for g3 in N do if g1* f(t)
*f(t^y)*f(t^(y^2))* f((t^(y^2))^x) eq g2*f(t)
*f(t^y)*f(t^(y^2))*f(t)*g3
then print g1, g2, g3; end if;
end for; end for; end for;
for g1 in N do for g2 in N do for g3 in N do if g1*
f((t^(y^2))^x)*f(t) *f((t^(y^2))^x) eq g2*f(t)
*f((t^(y^2))^x) *g3
then print g1, g2, g3; end if;
end for; end for; end for;
for g1 in N do for g2 in N do for g3 in N do if
g1*f(t)*f(t^y) *f(t^(y^2)) * f((t^(y^2))^x) eq
g2*f(t)*f(t^y) *g3 then print g1, g2, g3; end if;
end for; end for; end for;
print f(t);
(1, 2)(3, 5)(4, 7)(6, 9)(8, 14)(12, 19)(13, 20)(15, 22)(16,
24)(18, 25)(21, 31)(23, 35)(26, 28)(27, 41)(29, 43)(30,
44)(32, 46)(33, 47)(34, 39)(36, 48)(37, 50)(38, 51)(40,
45)(42, 52)(49, 54)(53, 55)
print f(t^y);
(1, 3)(2, 11)(4, 8)(6, 16)(7, 26)(9, 28)(10, 21)(13,
36)(14, 24)(17, 37)(18, 33)(19, 50)(20, 53)(22, 31)(23,
```

```
45)(25, 55)(27, 29)(30, 43)(32, 49)(34, 52)(35, 51)(38,
46)(39, 44)(40, 54)(41, 42)(47, 48)
print f(t^(y^2));
(1, 4)(2, 10)(3, 12)(5, 23)(6, 13)(9, 35)(11, 27)(14,
44)(15, 38)(16, 29)(17, 30)(19, 20)(21, 36)(22, 53)(24,
55)(25, 41)(26, 46)(28, 47)(31, 49)(32, 52)(33, 45)(34,
50)(37, 48)(39, 54)(40, 42)(43, 51)
print f((t^(y^2))^x);
(1, 6)(2, 17)(3, 15)(4, 18)(5, 39)(7, 34)(8, 49)(10,
40)(11, 32)(12, 42)(14, 53)(19, 55)(20, 46)(21, 47)(22,
25)(23, 43)(24, 45)(26, 48)(27, 51)(28, 41)(29, 50)(30,
38)(31, 35)(33, 37)(36, 44)(52, 54)
A:=Sym(4);
x:=A!(4,1,2);
y:=A!(4,1,3);
A4:=sub<A|x,y>;
print Order(A4);
12
N:=A4;
N0:=Stabiliser(N,4);
N01:=Stabiliser(N0,1);
print Orbits(N01);
[
    GSet{ 1 },
    GSet{ 2 },
    GSet{ 3 },
    GSet{ 4 }
]
print Order(N01);
1
Transitivity(A4);
2
{[4,1,2]}^A4;
{[4,1,3]}^A4;
S4:=Sym(4);
x:=S4!(1,2,3);
y:=S4!(4,1,2);
A4:=sub<S4|x,y>;
N:=sub<A4|x,y>;
C:=Centralizer(N, y^-1);
T:=Transversal(N,C);
print #T;
4
```

```
for i in [1..4] do print (y^-1)^T[i],
[4,1,2,4,1,2,4,1,2,4,1]^T[i]; end for;
(1, 4, 2)
[ 4, 1, 2, 4, 1, 2, 4, 1, 2, 4, 1 ]
(2, 4, 3)
[ 4, 2, 3, 4, 2, 3, 4, 2, 3, 4, 2 ]
(1, 3, 4)
[ 4, 3, 1, 4, 3, 1, 4, 3, 1, 4, 3 ]
(1, 2, 3)
[ 2, 1, 3, 2, 1, 3, 2, 1, 3, 2, 1 ]
C:=Centralizer(N, x*y);
T:=Transversal(N,C);
print #T;
3
for i in [1..3] do print (x*y)^T[i], [4,1,4,1,4]^T[i]; end
for;
(1, 4)(2, 3)
[ 4, 1, 4, 1, 4 ]
(1, 3)(2, 4)
[ 4, 2, 4, 2, 4 ]
(1, 2)(3, 4)
[ 4, 3, 4, 3, 4 ]
C:=Centralizer(N, y);
T:=Transversal(N,C);
print #T;
4
for i in [1..4] do print (y)^T[i],
[3,4,3,2,3,4,3,2,3,4]^T[i]; end for;
(1, 2, 4)
[ 3, 4, 3, 2, 3, 4, 3, 2, 3, 4 ]
(2, 3, 4)
[ 1, 4, 1, 3, 1, 4, 1, 3, 1, 4 ]
(1, 4, 3)
[ 2, 4, 2, 1, 2, 4, 2, 1, 2, 4 ]
(1, 3, 2)
[ 4, 2, 4, 3, 4, 2, 4, 3, 4, 2 ]

{[1,3,4,2,1,3], [2,4,3,1,2,4]}^A4;
GSet{
    {
        [ 4, 2, 1, 3, 4, 2 ],
        [ 3, 1, 2, 4, 3, 1 ]
    },
    {
```

```
        [ 1, 4, 2, 3, 1, 4 ],
        [ 3, 2, 4, 1, 3, 2 ]
    },
    {
        [ 2, 1, 4, 3, 2, 1 ],
        [ 3, 4, 1, 2, 3, 4 ]
    },
    {
        [ 1, 2, 3, 4, 1, 2 ],
        [ 4, 3, 2, 1, 4, 3 ]
    },
    {
        [ 2, 4, 3, 1, 2, 4 ],
        [ 1, 3, 4, 2, 1, 3 ]
    },
    {
        [ 4, 1, 3, 2, 4, 1 ],
        [ 2, 3, 1, 4, 2, 3 ]
    }
}

S4:=Sym(4);
x:=S4!(1,2,3);
y:=S4!(4,1,2);
A4:=sub<S4|x,y>;
N:=sub<A4|x,y>;
C:=Centralizer(N, y^-1);
T:=Transversal(N,C);
for i in [1..4] do print (y^-1)^T[i],
[3,4,3,1,3,2,3,4,3,1]^T[i]; end for;
(1, 4, 2)
[ 3, 4, 3, 1, 3, 2, 3, 4, 3, 1 ]
(2, 4, 3)
[ 1, 4, 1, 2, 1, 3, 1, 4, 1, 2 ]
(1, 3, 4)
[ 2, 4, 2, 3, 2, 1, 2, 4, 2, 3 ]
(1, 2, 3)
[ 4, 2, 4, 1, 4, 3, 4, 2, 4, 1 ]
```

Magma work for the Group $L_2(23)$ over S_4

```
S4:=Sym(4);
a:=S4!(4,1,2,3);
b:=S4!(2,3);
S4:=sub<S4|a,b>;
```

```
G<x,y,t>:=Group<x,y,t|x^4,y^2,(x*y)^3,t^2, (t,y), (t^x,y),
(x*t)^11, (((x*y)^y)*t)^11, (y^(x^2)*y*t)^12,
(y^(x^2)*t)^12,(y^(x^2)*t*t^(x^2))^11,y*((t^x*t)^3)>;
print Order(G);
6072
print Index(G, sub<G|x,y>:CosetLimit:=5000000,Hard:=true);
253
N:=sub<S4|x,y>;
C:=Centralizer(N,y);
T:=Transversal(N,C);
print #T;
6
for i in [1..6] do print (y)^T[i], [4,1,4,1,4,1]^T[i]; end
for;
(2, 3)
[ 4, 1, 4, 1, 4, 1 ]
(3, 4)
[ 1, 2, 1, 2, 1, 2 ]
(1, 4)
[ 2, 3, 2, 3, 2, 3 ]
(2, 4)
[ 1, 3, 1, 3, 1, 3 ]
(1, 2)
[ 3, 4, 3, 4, 3, 4 ]
(1, 3)
[ 2, 4, 2, 4, 2, 4 ]
print S4;
Symmetric group S4 acting on a set of cardinality 4
Order = 24 = 2^3 * 3
    (1, 2, 3, 4)
    (1, 2)
N4:=Stabiliser(S4,4);
N41:=Stabiliser(N4,1);
print N41;
Permutation group N41 acting on a set of cardinality 4
Order = 2
 (2, 3)
for g in S4 do if 1^g eq 4 and 4^g eq 1 then
N41:=sub<S4|N41,g>; end if; end\
 for;
print Orbits(N41);
[
    GSet{ 1, 4 },
    GSet{ 2, 3 }
```

```
]
N4:=Stabiliser(S4,4);
N41:=Stabiliser(N4,1);
N412:=Stabiliser(N41,2);
print N412;
Permutation group N412 acting on a set of cardinality 4
Order = 1
for g in S4 do if 1^g eq 4 and 4^g eq 1 then
N412:=sub<S4|N412,g>; end if; end for;
print Orbits(N412);
[
    GSet{ 1, 4 },
    GSet{ 2, 3 }
]
f, G1, k := CosetAction(G, sub<G|x,y>);
N:=sub<G1|f(x),f(y)>;
for g2 in N do for g3 in N do if  f(t) *f(t^x)*f(t^(x^2))*
f(t^(x^3)) * f(t^(x^2))   eq  g2*(f(t)*f(t^x)* f(t^(x^2)) *
f(t^x) )^g3 then print g2, g3;  end if; end for; end for;
(2, 5)(3, 4)(6, 15)(7, 19)(8, 17)(9, 10)(11, 12)(13,
14)(16, 39)(18, 30)(20, 28)(21, 48)(22, 50)(23, 55)(24,
42)(25, 26)(29, 53)(31, 32)(33, 34)(35, 36)(37, 38)(40,
46)(41, 93)(43, 73)(45, 100)(47, 105)(49, 65)(51, 112)(52,
115)(54, 82)(56, 80)(57, 120)(58, 122)(59, 127)(60,
117)(61, 113)(62, 63)(66, 102)(67, 68)(69, 130)(70, 71)(72,
125)(74, 96)(75, 76)(77, 78)(81, 133)(83, 84)(85, 86)(87,
88)(89, 90)(91, 92)(94, 119)(95, 183)(98, 99)(101,
158)(103, 104)(106, 161)(107, 108)(110, 111)(114, 207)(116,
211)(118, 141)(121, 145)(123, 220)(124, 222)(126, 186)(128,
225)(129, 228)(131, 232)(132, 234)(134, 236)(135, 208)(136,
137)(138, 168)(139, 191)(140, 157)(142, 187)(143, 229)(144,
167)(146, 221)(147, 243)(148, 226)(149, 244)(150, 240)(151,
185)(152, 153)(154, 223)(155, 156)(159, 237)(160, 230)(162,
213)(163, 164)(165, 166)(169, 239)(170, 171)(172, 173)(174,
241)(175, 176)(177, 178)(179, 180)(181, 182)(184, 224)(188,
189)(190, 194)(192, 193)(195, 196)(197, 198)(199, 200)(201,
202)(203, 204)(205, 206)(209, 210)(212, 231)(214, 215)(216,
217)(218, 219)(227, 253)(235, 249)(238, 250)(245, 252)(246,
247)(248, 251)
Id(N)
f,G1,k:=CosetAction(G,sub<G|x,y>);
t0:=f(t);
t1:=f(t^x);
t2:=f(t^(x^2));
```

```
t3:=f(t^(x^3));
print t0, t1, t2, t3;
G<x,y,t>:=Group<x,y,t|x^4,y^2,(x*y)^3,t^2, (t,y), (t^x,y),
(x*t)^11, (((x*y)^y)*t)^11, (y^(x^2)*y*t)^12,
(y^(x^2)*t)^12,(y^(x^2)*t*t^(x^2))^11,y*((t^x*t)^3)>;
f,G1,k:=CosetAction(G,sub<G|x,y>);
S4:=Sym(4);
a:=S4!(1,2,3,4);
b:=S4!(2,3);
N:=S4;
prodim := function(pt, Q, I)
/*
Return the image of pt under permutations Q[I] applied
sequentially.
*/
  v := pt;
   for i in I do
       v := v^(Q[i]);
   end for;
return v;
end function;
     ts := [ (t^(x^i)) @ f : i in [1 .. 4] ];
     cst := [null : i in [1 .. 253]] where null is
[Integers() | ];
      for i := 1 to 4 do
         cst[prodim(1, ts, [i])] := [i];
      end for;
      for i := 1 to 4 do
       for j in {1, 2, 3, 4} diff {i} do
         cst[prodim(1, ts, [i, j])] := [i, j];
         end for;
    end for;
for i := 1 to 4 do
       for j in {1, 2, 3, 4} diff {i} do
          for k in {1,2,3,4} diff {i,j} do
         cst[prodim(1, ts, [i, j,k])] := [i, j,k];
         end for;
    end for;
end for;
for i := 1 to 4 do
       for j in {1, 2, 3, 4} diff {i} do
          for k in {1,2,3,4} diff {i,j} do
            for l in {1,2,3,4} diff {i,j,k} do
         cst[prodim(1, ts, [i, j,k,l])] := [i, j,k,l];
```

```
        end for;
    end for;
end for;
end for;
for i := 1 to 4 do
       for j in {1, 2, 3, 4} diff {i} do
          for k in {1,2,3,4} diff {i,j} do
            for l in {1,2,3,4} diff {i,j,k} do
         cst[prodim(1, ts, [i, j,k,l,i])] := [i, j,k,l,i];
         end for;
    end for;
end for;
end for;
for i := 1 to 4 do
       for j in {1, 2, 3, 4} diff {i} do
          for k in {1,2,3,4} diff {i,j} do
            for l in {1,2,3,4} diff {i,j,k} do
         cst[prodim(1, ts, [i, j,k,l,j])] := [i, j,k,l,j];
         end for;
    end for;
end for;
end for;
for i := 1 to 4 do
       for j in {1, 2, 3, 4} diff {i} do
          for k in {1,2,3,4} diff {i,j} do
         cst[prodim(1, ts, [i, j,k,i,j])] := [i, j,k,i,j];
         end for;
    end for;
end for;
for i := 1 to 4 do
       for j in {1, 2, 3, 4} diff {i} do
          for k in {1,2,3,4} diff {i,j} do
         cst[prodim(1, ts, [i, j,k,i,k])] := [i, j,k,i,k];
         end for;
    end for;
end for;
for i := 1 to 4 do
       for j in {1, 2, 3, 4} diff {i} do
          for k in {1,2,3,4} diff {i,j} do
            for l in {1,2,3,4} diff {i,j,k} do
         cst[prodim(1, ts, [i, j,k,i,l])] := [i, j,k,i,l];
         end for;
    end for;
end for;
```

```
end for;
for i := 1 to 4 do
       for j in {1, 2, 3, 4} diff {i} do
          for k in {1,2,3,4} diff {i,j} do
         cst[prodim(1, ts, [i, j,k,i])] := [i, j,k,i];
         end for;
    end for;
end for;
for i := 1 to 4 do
       for j in {1, 2, 3, 4} diff {i} do
          for k in {1,2,3,4} diff {i,j} do
         cst[prodim(1, ts, [i, j,k,i,j])] := [i, j,k,i,j];
         end for;
    end for;
end for;
for i := 1 to 4 do
       for j in {1, 2, 3, 4} diff {i} do
          for k in {1,2,3,4} diff {i,j} do
         cst[prodim(1, ts, [i, j,k,i,l])] := [i, j,k,i,l];
         end for;
    end for;
end for;
cst[prodim(1, ts, [4,1,4])] := [4,1,4];cst[prodim(1, ts,
[4,2,4])] := [4,2,4];
cst[prodim(1, ts, [4,3,4])] := [4,3,4];
cst[prodim(1, ts, [1,2,1])] := [1,2,1];
cst[prodim(1, ts, [1,3,1])] := [1,3,1];
cst[prodim(1, ts, [2,3,2])] := [2,3,2];
cst[prodim(1, ts, [4,1,4,2])] := [4,1,4,2];
cst[prodim(1, ts, [4,1,4,3])] := [4,1,4,3];
cst[prodim(1, ts, [4,2,4,1])] := [4,2,4,1];
cst[prodim(1, ts, [4,2,4,3])] := [4,2,4,3];
cst[prodim(1, ts, [4,3,4,1])] := [4,3,4,1];
cst[prodim(1, ts, [4,3,4,2])] := [4,3,4,2];
cst[prodim(1, ts, [1,2,1,4])] := [1,2,1,4];
cst[prodim(1, ts, [1,2,1,3])] := [1,2,1,3];
cst[prodim(1, ts, [1,3,1,4])] := [1,3,1,4];
cst[prodim(1, ts, [1,3,1,2])] := [1,3,1,2];
cst[prodim(1, ts, [2,3,2,4])] := [2,3,2,4];
cst[prodim(1, ts, [2,3,2,1])] := [2,3,2,1];
cst[prodim(1, ts, [4,1,2,1])] := [4,1,2,1];
cst[prodim(1, ts, [4,2,1,2])] := [4,2,1,2];
cst[prodim(1, ts, [4,1,3,1])] := [4,1,3,1];
cst[prodim(1, ts, [4,3,1,3])] := [4,3,1,3];
```

```
cst[prodim(1, ts, [4,2,3,2])] := [4,2,3,2];
cst[prodim(1, ts, [4,3,2,3])] := [4,3,2,3];
cst[prodim(1, ts, [1,4,2,4])] := [1,4,2,4];
cst[prodim(1, ts, [1,2,4,2])] := [1,2,4,2];
cst[prodim(1, ts, [1,4,3,4])] := [1,4,3,4];
cst[prodim(1, ts, [1,3,4,3])] := [1,3,4,3];
cst[prodim(1, ts, [2,4,1,4])] := [2,4,1,4];
cst[prodim(1, ts, [2,1,4,1])] := [2,1,4,1];
cst[prodim(1, ts, [4,1,2,4,3,2])] := [4,1,2,4,3,2];
cst[prodim(1, ts, [4,1,3,4,2,3])] := [4,1,3,4,2,3];
cst[prodim(1, ts, [4,3,2,4,1,2])] := [4,3,2,4,1,2];
cst[prodim(1, ts, [4,2,1,4,3,1])] := [4,2,1,4,3,1];
cst[prodim(1, ts, [4,3,1,4,2,1])] := [4,3,1,4,2,1];
cst[prodim(1, ts, [4,2,3,4,1,3])] := [4,2,3,4,1,3];
cst[prodim(1, ts, [4,1,2,3,1,4])] := [4,1,2,3,1,4];
cst[prodim(1, ts, [4,2,3,1,2,4])] := [4,2,3,1,2,4];
cst[prodim(1, ts, [4,2,1,3,2,4])] := [4,2,1,3,2,4];
cst[prodim(1, ts, [4,1,3,2,1,4])] := [4,1,3,2,1,4];
cst[prodim(1, ts, [4,3,1,2,3,4])] := [4,3,1,2,3,4];
cst[prodim(1, ts, [4,3,2,1,3,4])] := [4,3,2,1,3,4];
cst[prodim(1, ts, [1,2,4,3,2,1])] := [1,2,4,3,2,1];
cst[prodim(1, ts, [1,3,4,2,3,1])] := [1,3,4,2,3,1];
for i in [1..253] do print i, cst[i]; end for;

f(t) *f(t^x)*f(t^(x^2))* f(t^( x^3)) * f(t^(x^2))   eq
f(((x^3)^y)^(y^(y^(x^3))))*(f(t)*f(t^x)*f(t^(x^2))*f(t^x))^
f((x^3));
true
```

Magma work for the Group $PGL_2(17)$ over A_4

```
S4:=Sym(4);
x:=S4!(1,2,3);
y:=S4!(4,1,2);
G<x,y,t>:=Group<x,y,t|x^3,y^3,(x*y)^2,t^2, (t,x),(y*t)^18 ,
(x*y*t)^16 ,(y*t* (t^(y^2))^x)^4 , (x*y*t*t^(y^2))^9>;
print Order(G);
 4896
print Index(G, sub<G|x,y>:CosetLimit:=5000000,Hard:=true);
 408
f, G1, k := CosetAction(G, sub<G|x,y>);
N:=sub<G1|f(x),f(y)>;
{[4,1,2]}^A4;
{[4,1,3]}^A4;
{[4,1,4,2]}^A4;
```

```
{[4,1,4,3]}^A4;
{[4,1,2,3]}^A4;
{[4,1,3,2]}^A4;
{[4,1,4,1,2]}^A4;
{[4,1,4,1,3]}^A4;
{[4,1,4,1,4,2]}^A4;
{[4,1,4,1,4,3]}^A4;
{[4,1,4,1,4,1,2]}^A4;
{[4,1,4,1,4,1,3]}^A4;
S4:=Sym(4);
x:=S4!(1,2,3);
y:=S4!(4,1,2);
A4:=sub<S4|x,y>;
N:=sub<A4|x,y>;
C:=Centralizer(N, y);
T:=Transversal(N,C);
print #T;
4
for i in [1..4] do print (y)^T[i], [3,4,3,1,3,2,3,4]^T[i];
end for;
(1, 2, 4)
[ 3, 4, 3, 1, 3, 2, 3, 4 ]
(2, 3, 4)
[ 1, 4, 1, 2, 1, 3, 1, 4 ]
(1, 4, 3)
[ 2, 4, 2, 3, 2, 1, 2, 4 ]
(1, 3, 2)
[ 4, 2, 4, 1, 4, 3, 4, 2 ]
> f(x^-1)*f(t^y)*f(t)*f(t^(y^2))*f(t) eq
f(t)*f(t^y)*f(t)*f((t^(y^2))^x);
true
f(x^-1)*f(t^y)*f(t)*f(t^(y^2))*f(t)*f(t^y) eq
f(t)*f(t^y)*f(t)*f((t^(y^2))^
x)*f(t^y);
true
f((((y^-1)^(x^-1))^ ((x*y)^y))^-1) *f(t^(y^2)) *f(t)*f(t^y)
*f((t^(y^2))^x)*f(t^y) eq
f(t)*f(t^y)*f(t)*f(t^y)*f(t^(y^2));
true

prodim := function(pt, Q, I)
 /*
Return the image of pt under permutations Q[I] applied
sequentially.
```

```
 */
v := pt;
for i in I do
v := v^(Q[i]);
end for;
return v;
end function;
G<x,y,t>:=Group<x,y,t|x^3,y^3,(x*y)^2,t^2, (t,x),(y*t)^18 ,
(x*y*t)^16 ,(y*t* (t^(y^2))^x)^4 , (x*y*t*t^(y^2))^9>;
f, G1, k := CosetAction(G, sub<G|x,y>);
N:=sub<G1|f(x),f(y)>;
ts := [ Id(G1) : i in [1 .. 4] ];
ts[1]:=f(t^y);
ts[2]:=f(t^(y^2));
ts[3]:=f((t^(y^2))^x);
ts[4]:=f(t^(y^3));
cst := [null : i in [1 .. 408]] where null is [Integers() |
];
 for i := 1 to 4 do
        cst[prodim(1, ts, [i])] := [i];
     end for;

 for i := 1 to 4 do
     for j in {1, 2, 3, 4} diff {i} do
      cst[prodim(1, ts, [ i, j ])] := [ i, j ];
      cst[prodim(1, ts, [ i, j, i ])] := [ i, j, i ];
      cst[prodim(1, ts, [ i, j, i, j ])] := [ i, j, i, j ];
      cst[prodim(1, ts, [ i, j, i, j, i ])] := [ i, j, i,
j, i ];
      cst[prodim(1, ts, [ i, j, i, j, i, j ])] := [ i, j,
i, j, i, j ];
       end for;
   end for;
cst[prodim(1, ts, [ 4, 1, 2 ])] := [ 4, 1, 2 ];
cst[prodim(1, ts, [ 4, 2, 3 ])] := [ 4, 2, 3 ];
cst[prodim(1, ts, [ 1, 4, 3 ])] := [ 1, 4, 3 ];
cst[prodim(1, ts, [ 3, 2, 1 ])] := [ 3, 2, 1 ];
cst[prodim(1, ts, [ 2, 1, 3 ])] := [ 2, 1, 3 ];
cst[prodim(1, ts, [ 3, 4, 2 ])] := [ 3, 4, 2 ];
cst[prodim(1, ts, [ 1, 3, 2 ])] := [ 1, 3, 2 ];
cst[prodim(1, ts, [ 3, 1, 4 ])] := [ 3, 1, 4 ];
cst[prodim(1, ts, [ 2, 4, 1 ])] := [ 2, 4, 1 ];
cst[prodim(1, ts, [ 4, 3, 1 ])] := [ 4, 3, 1 ];
cst[prodim(1, ts, [ 2, 3, 4 ])] := [ 2, 3, 4 ];
```

```
cst[prodim(1, ts, [ 1, 2, 4 ])] := [ 1, 2, 4 ];
cst[prodim(1, ts, [ 4, 1, 3 ])] := [ 4, 1, 3 ];
cst[prodim(1, ts, [ 1, 2, 3 ])] := [ 1, 2, 3 ];
cst[prodim(1, ts, [ 3, 4, 1 ])] := [ 3, 4, 1 ];
cst[prodim(1, ts, [ 4, 2, 1 ])] := [ 4, 2, 1 ];
cst[prodim(1, ts, [ 2, 4, 3 ])] := [ 2, 4, 3 ];
cst[prodim(1, ts, [ 2, 1, 4 ])] := [ 2, 1, 4 ];
cst[prodim(1, ts, [ 1, 4, 2 ])] := [ 1, 4, 2 ];
cst[prodim(1, ts, [ 1, 3, 4 ])] := [ 1, 3, 4 ];
cst[prodim(1, ts, [ 2, 3, 1 ])] := [ 2, 3, 1 ];
cst[prodim(1, ts, [ 3, 2, 4 ])] := [ 3, 2, 4 ];
cst[prodim(1, ts, [ 4, 3, 2 ])] := [ 4, 3, 2 ];
cst[prodim(1, ts, [ 3, 1, 2 ])] := [ 3, 1, 2 ];
cst[prodim(1, ts, [ 4, 1, 4, 2 ])] := [ 4, 1, 4, 2 ];
cst[prodim(1, ts, [ 1, 2, 1, 4 ])] := [ 1, 2, 1, 4 ];
cst[prodim(1, ts, [ 3, 1, 3, 4 ])] := [ 3, 1, 3, 4 ];
cst[prodim(1, ts, [ 1, 4, 1, 3 ])] := [ 1, 4, 1, 3 ];
cst[prodim(1, ts, [ 1, 3, 1, 2 ])] := [ 1, 3, 1, 2 ];
cst[prodim(1, ts, [ 2, 3, 2, 4 ])] := [ 2, 3, 2, 4 ];
cst[prodim(1, ts, [ 3, 4, 3, 2 ])] := [ 3, 4, 3, 2 ];
cst[prodim(1, ts, [ 4, 2, 4, 3 ])] := [ 4, 2, 4, 3 ];
cst[prodim(1, ts, [ 3, 2, 3, 1 ])] := [ 3, 2, 3, 1 ];
cst[prodim(1, ts, [ 4, 3, 4, 1 ])] := [ 4, 3, 4, 1 ];
cst[prodim(1, ts, [ 2, 1, 2, 3 ])] := [ 2, 1, 2, 3 ];
cst[prodim(1, ts, [ 2, 4, 2, 1 ])] := [ 2, 4, 2, 1 ];
cst[prodim(1, ts, [ 4, 1, 4, 3 ])] := [ 4, 1, 4, 3 ];
cst[prodim(1, ts, [ 2, 1, 2, 4 ])] := [ 2, 1, 2, 4 ];
cst[prodim(1, ts, [ 4, 3, 4, 2 ])] := [ 4, 3, 4, 2 ];
cst[prodim(1, ts, [ 2, 4, 2, 3 ])] := [ 2, 4, 2, 3 ];
cst[prodim(1, ts, [ 1, 4, 1, 2 ])] := [ 1, 4, 1, 2 ];
cst[prodim(1, ts, [ 2, 3, 2, 1 ])] := [ 2, 3, 2, 1 ];
cst[prodim(1, ts, [ 3, 1, 3, 2 ])] := [ 3, 1, 3, 2 ];
cst[prodim(1, ts, [ 1, 3, 1, 4 ])] := [ 1, 3, 1, 4 ];
cst[prodim(1, ts, [ 1, 2, 1, 3 ])] := [ 1, 2, 1, 3 ];
cst[prodim(1, ts, [ 4, 2, 4, 1 ])] := [ 4, 2, 4, 1 ];
cst[prodim(1, ts, [ 3, 2, 3, 4 ])] := [ 3, 2, 3, 4 ];
cst[prodim(1, ts, [ 3, 4, 3, 1 ])] := [ 3, 4, 3, 1 ];
cst[prodim(1, ts, [ 4, 1, 2, 4 ])] := [ 4, 1, 2, 4 ];
cst[prodim(1, ts, [ 4, 2, 3, 4 ])] := [ 4, 2, 3, 4 ];
cst[prodim(1, ts, [ 1, 4, 3, 1 ])] := [ 1, 4, 3, 1 ];
cst[prodim(1, ts, [ 3, 2, 1, 3 ])] := [ 3, 2, 1, 3 ];
cst[prodim(1, ts, [ 2, 1, 3, 2 ])] := [ 2, 1, 3, 2 ];
cst[prodim(1, ts, [ 3, 4, 2, 3 ])] := [ 3, 4, 2, 3 ];
cst[prodim(1, ts, [ 1, 3, 2, 1 ])] := [ 1, 3, 2, 1 ];
```

```
cst[prodim(1, ts, [ 3, 1, 4, 3 ])] := [ 3, 1, 4, 3 ];
cst[prodim(1, ts, [ 2, 4, 1, 2 ])] := [ 2, 4, 1, 2 ];
cst[prodim(1, ts, [ 4, 3, 1, 4 ])] := [ 4, 3, 1, 4 ];
cst[prodim(1, ts, [ 2, 3, 4, 2 ])] := [ 2, 3, 4, 2 ];
cst[prodim(1, ts, [ 1, 2, 4, 1 ])] := [ 1, 2, 4, 1 ];
cst[prodim(1, ts, [ 4, 1, 3, 4 ])] := [ 4, 1, 3, 4 ];
cst[prodim(1, ts, [ 1, 2, 3, 1 ])] := [ 1, 2, 3, 1 ];
cst[prodim(1, ts, [ 3, 4, 1, 3 ])] := [ 3, 4, 1, 3 ];
cst[prodim(1, ts, [ 4, 2, 1, 4 ])] := [ 4, 2, 1, 4 ];
cst[prodim(1, ts, [ 2, 4, 3, 2 ])] := [ 2, 4, 3, 2 ];
cst[prodim(1, ts, [ 2, 1, 4, 2 ])] := [ 2, 1, 4, 2 ];
cst[prodim(1, ts, [ 1, 4, 2, 1 ])] := [ 1, 4, 2, 1 ];
cst[prodim(1, ts, [ 1, 3, 4, 1 ])] := [ 1, 3, 4, 1 ];
cst[prodim(1, ts, [ 2, 3, 1, 2 ])] := [ 2, 3, 1, 2 ];
cst[prodim(1, ts, [ 3, 2, 4, 3 ])] := [ 3, 2, 4, 3 ];
cst[prodim(1, ts, [ 4, 3, 2, 4 ])] := [ 4, 3, 2, 4 ];
cst[prodim(1, ts, [ 3, 1, 2, 3 ])] := [ 3, 1, 2, 3 ];
cst[prodim(1, ts, [ 4, 1, 2, 3 ])] := [ 4, 1, 2, 3 ];
cst[prodim(1, ts, [ 4, 2, 3, 1 ])] := [ 4, 2, 3, 1 ];
cst[prodim(1, ts, [ 1, 4, 3, 2 ])] := [ 1, 4, 3, 2 ];
cst[prodim(1, ts, [ 3, 2, 1, 4 ])] := [ 3, 2, 1, 4 ];
cst[prodim(1, ts, [ 2, 1, 3, 4 ])] := [ 2, 1, 3, 4 ];
cst[prodim(1, ts, [ 3, 4, 2, 1 ])] := [ 3, 4, 2, 1 ];
cst[prodim(1, ts, [ 1, 3, 2, 4 ])] := [ 1, 3, 2, 4 ];
cst[prodim(1, ts, [ 3, 1, 4, 2 ])] := [ 3, 1, 4, 2 ];
cst[prodim(1, ts, [ 2, 4, 1, 3 ])] := [ 2, 4, 1, 3 ];
cst[prodim(1, ts, [ 4, 3, 1, 2 ])] := [ 4, 3, 1, 2 ];
cst[prodim(1, ts, [ 2, 3, 4, 1 ])] := [ 2, 3, 4, 1 ];
cst[prodim(1, ts, [ 1, 2, 4, 3 ])] := [ 1, 2, 4, 3 ];
cst[prodim(1, ts, [ 4, 1, 3, 2 ])] := [ 4, 1, 3, 2 ];
cst[prodim(1, ts, [ 1, 2, 3, 4 ])] := [ 1, 2, 3, 4 ];
cst[prodim(1, ts, [ 3, 4, 1, 2 ])] := [ 3, 4, 1, 2 ];
cst[prodim(1, ts, [ 4, 2, 1, 3 ])] := [ 4, 2, 1, 3 ];
cst[prodim(1, ts, [ 2, 4, 3, 1 ])] := [ 2, 4, 3, 1 ];
cst[prodim(1, ts, [ 2, 1, 4, 3 ])] := [ 2, 1, 4, 3 ];
cst[prodim(1, ts, [ 1, 4, 2, 3 ])] := [ 1, 4, 2, 3 ];
cst[prodim(1, ts, [ 1, 3, 4, 2 ])] := [ 1, 3, 4, 2 ];
cst[prodim(1, ts, [ 2, 3, 1, 4 ])] := [ 2, 3, 1, 4 ];
cst[prodim(1, ts, [ 3, 2, 4, 1 ])] := [ 3, 2, 4, 1 ];
cst[prodim(1, ts, [ 4, 3, 2, 1 ])] := [ 4, 3, 2, 1 ];
cst[prodim(1, ts, [ 3, 1, 2, 4 ])] := [ 3, 1, 2, 4 ];
cst[prodim(1, ts, [ 4, 1, 4, 1, 3 ])] := [ 4, 1, 4, 1, 3 ];
cst[prodim(1, ts, [ 1, 4, 1, 4, 2 ])] := [ 1, 4, 1, 4, 2 ];
cst[prodim(1, ts, [ 3, 1, 3, 1, 2 ])] := [ 3, 1, 3, 1, 2 ];
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```
cst[prodim(1, ts, [ 3, 2, 3, 2, 4 ])] := [ 3, 2, 3, 2, 4 ];
cst[prodim(1, ts, [ 3, 4, 3, 4, 1 ])] := [ 3, 4, 3, 4, 1 ];
cst[prodim(1, ts, [ 4, 2, 4, 2, 1 ])] := [ 4, 2, 4, 2, 1 ];
cst[prodim(1, ts, [ 1, 2, 1, 2, 3 ])] := [ 1, 2, 1, 2, 3 ];
cst[prodim(1, ts, [ 2, 4, 2, 4, 3 ])] := [ 2, 4, 2, 4, 3 ];
cst[prodim(1, ts, [ 4, 3, 4, 3, 2 ])] := [ 4, 3, 4, 3, 2 ];
cst[prodim(1, ts, [ 2, 3, 2, 3, 1 ])] := [ 2, 3, 2, 3, 1 ];
cst[prodim(1, ts, [ 2, 1, 2, 1, 4 ])] := [ 2, 1, 2, 1, 4 ];
cst[prodim(1, ts, [ 1, 3, 1, 3, 4 ])] := [ 1, 3, 1, 3, 4 ];
cst[prodim(1, ts, [ 4, 1, 4, 2, 1 ])] := [ 4, 1, 4, 2, 1 ];
cst[prodim(1, ts, [ 1, 2, 1, 4, 2 ])] := [ 1, 2, 1, 4, 2 ];
cst[prodim(1, ts, [ 3, 1, 3, 4, 1 ])] := [ 3, 1, 3, 4, 1 ];
cst[prodim(1, ts, [ 1, 4, 1, 3, 4 ])] := [ 1, 4, 1, 3, 4 ];
cst[prodim(1, ts, [ 1, 3, 1, 2, 3 ])] := [ 1, 3, 1, 2, 3 ];
cst[prodim(1, ts, [ 2, 3, 2, 4, 3 ])] := [ 2, 3, 2, 4, 3 ];
cst[prodim(1, ts, [ 3, 4, 3, 2, 4 ])] := [ 3, 4, 3, 2, 4 ];
cst[prodim(1, ts, [ 4, 2, 4, 3, 2 ])] := [ 4, 2, 4, 3, 2 ];
cst[prodim(1, ts, [ 3, 2, 3, 1, 2 ])] := [ 3, 2, 3, 1, 2 ];
cst[prodim(1, ts, [ 4, 3, 4, 1, 3 ])] := [ 4, 3, 4, 1, 3 ];
cst[prodim(1, ts, [ 2, 1, 2, 3, 1 ])] := [ 2, 1, 2, 3, 1 ];
cst[prodim(1, ts, [ 2, 4, 2, 1, 4 ])] := [ 2, 4, 2, 1, 4 ];
cst[prodim(1, ts, [ 4, 1, 4, 2, 3 ])] := [ 4, 1, 4, 2, 3 ];
cst[prodim(1, ts, [ 1, 2, 1, 4, 3 ])] := [ 1, 2, 1, 4, 3 ];
cst[prodim(1, ts, [ 3, 1, 3, 4, 2 ])] := [ 3, 1, 3, 4, 2 ];
cst[prodim(1, ts, [ 1, 4, 1, 3, 2 ])] := [ 1, 4, 1, 3, 2 ];
cst[prodim(1, ts, [ 1, 3, 1, 2, 4 ])] := [ 1, 3, 1, 2, 4 ];
cst[prodim(1, ts, [ 2, 3, 2, 4, 1 ])] := [ 2, 3, 2, 4, 1 ];
cst[prodim(1, ts, [ 3, 4, 3, 2, 1 ])] := [ 3, 4, 3, 2, 1 ];
cst[prodim(1, ts, [ 4, 2, 4, 3, 1 ])] := [ 4, 2, 4, 3, 1 ];
cst[prodim(1, ts, [ 3, 2, 3, 1, 4 ])] := [ 3, 2, 3, 1, 4 ];
cst[prodim(1, ts, [ 4, 3, 4, 1, 2 ])] := [ 4, 3, 4, 1, 2 ];
cst[prodim(1, ts, [ 2, 1, 2, 3, 4 ])] := [ 2, 1, 2, 3, 4 ];
cst[prodim(1, ts, [ 2, 4, 2, 1, 3 ])] := [ 2, 4, 2, 1, 3 ];
cst[prodim(1, ts, [ 4, 1, 4, 3, 1 ])] := [ 4, 1, 4, 3, 1 ];
cst[prodim(1, ts, [ 2, 1, 2, 4, 1 ])] := [ 2, 1, 2, 4, 1 ];
cst[prodim(1, ts, [ 4, 3, 4, 2, 3 ])] := [ 4, 3, 4, 2, 3 ];
cst[prodim(1, ts, [ 2, 4, 2, 3, 4 ])] := [ 2, 4, 2, 3, 4 ];
cst[prodim(1, ts, [ 1, 4, 1, 2, 4 ])] := [ 1, 4, 1, 2, 4 ];
cst[prodim(1, ts, [ 2, 3, 2, 1, 3 ])] := [ 2, 3, 2, 1, 3 ];
cst[prodim(1, ts, [ 3, 1, 3, 2, 1 ])] := [ 3, 1, 3, 2, 1 ];
cst[prodim(1, ts, [ 1, 3, 1, 4, 3 ])] := [ 1, 3, 1, 4, 3 ];
cst[prodim(1, ts, [ 1, 2, 1, 3, 2 ])] := [ 1, 2, 1, 3, 2 ];
cst[prodim(1, ts, [ 4, 2, 4, 1, 2 ])] := [ 4, 2, 4, 1, 2 ];
cst[prodim(1, ts, [ 3, 2, 3, 4, 2 ])] := [ 3, 2, 3, 4, 2 ];
```

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cst[prodim(1, ts, [ 3, 4, 3, 1, 4 ])] := [ 3, 4, 3, 1, 4 ];
cst[prodim(1, ts, [ 4, 1, 4, 3, 2 ])] := [ 4, 1, 4, 3, 2 ];
cst[prodim(1, ts, [ 2, 1, 2, 4, 3 ])] := [ 2, 1, 2, 4, 3 ];
cst[prodim(1, ts, [ 4, 3, 4, 2, 1 ])] := [ 4, 3, 4, 2, 1 ];
cst[prodim(1, ts, [ 2, 4, 2, 3, 1 ])] := [ 2, 4, 2, 3, 1 ];
cst[prodim(1, ts, [ 1, 4, 1, 2, 3 ])] := [ 1, 4, 1, 2, 3 ];
cst[prodim(1, ts, [ 2, 3, 2, 1, 4 ])] := [ 2, 3, 2, 1, 4 ];
cst[prodim(1, ts, [ 3, 1, 3, 2, 4 ])] := [ 3, 1, 3, 2, 4 ];
cst[prodim(1, ts, [ 1, 3, 1, 4, 2 ])] := [ 1, 3, 1, 4, 2 ];
cst[prodim(1, ts, [ 1, 2, 1, 3, 4 ])] := [ 1, 2, 1, 3, 4 ];
cst[prodim(1, ts, [ 4, 2, 4, 1, 3 ])] := [ 4, 2, 4, 1, 3 ];
cst[prodim(1, ts, [ 3, 2, 3, 4, 1 ])] := [ 3, 2, 3, 4, 1 ];
cst[prodim(1, ts, [ 3, 4, 3, 1, 2 ])] := [ 3, 4, 3, 1, 2 ];
cst[prodim(1, ts, [ 4, 1, 2, 4, 3 ])] := [ 4, 1, 2, 4, 3 ];
cst[prodim(1, ts, [ 4, 2, 3, 4, 1 ])] := [ 4, 2, 3, 4, 1 ];
cst[prodim(1, ts, [ 1, 4, 3, 1, 2 ])] := [ 1, 4, 3, 1, 2 ];
cst[prodim(1, ts, [ 3, 2, 1, 3, 4 ])] := [ 3, 2, 1, 3, 4 ];
cst[prodim(1, ts, [ 2, 1, 3, 2, 4 ])] := [ 2, 1, 3, 2, 4 ];
cst[prodim(1, ts, [ 3, 4, 2, 3, 1 ])] := [ 3, 4, 2, 3, 1 ];
cst[prodim(1, ts, [ 1, 3, 2, 1, 4 ])] := [ 1, 3, 2, 1, 4 ];
cst[prodim(1, ts, [ 3, 1, 4, 3, 2 ])] := [ 3, 1, 4, 3, 2 ];
cst[prodim(1, ts, [ 2, 4, 1, 2, 3 ])] := [ 2, 4, 1, 2, 3 ];
cst[prodim(1, ts, [ 4, 3, 1, 4, 2 ])] := [ 4, 3, 1, 4, 2 ];
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cst[prodim(1, ts, [ 1, 2, 4, 1, 3 ])] := [ 1, 2, 4, 1, 3 ];
cst[prodim(1, ts, [ 4, 1, 2, 3, 1 ])] := [ 4, 1, 2, 3, 1 ];
cst[prodim(1, ts, [ 4, 2, 3, 1, 2 ])] := [ 4, 2, 3, 1, 2 ];
cst[prodim(1, ts, [ 1, 4, 3, 2, 4 ])] := [ 1, 4, 3, 2, 4 ];
cst[prodim(1, ts, [ 3, 2, 1, 4, 2 ])] := [ 3, 2, 1, 4, 2 ];
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cst[prodim(1, ts, [ 2, 1, 3, 4, 3 ])] := [ 2, 1, 3, 4, 3 ];
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cst[prodim(1, ts, [ 1, 3, 2, 4, 2 ])] := [ 1, 3, 2, 4, 2 ];
```

```
cst[prodim(1, ts, [ 3, 1, 4, 2, 4 ])] := [ 3, 1, 4, 2, 4 ];
cst[prodim(1, ts, [ 2, 4, 1, 3, 1 ])] := [ 2, 4, 1, 3, 1 ];
cst[prodim(1, ts, [ 4, 3, 1, 2, 1 ])] := [ 4, 3, 1, 2, 1 ];
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cst[prodim(1, ts, [ 3, 4, 3, 4, 3, 2 ])] := [ 3, 4, 3, 4,
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cst[prodim(1, ts, [ 1, 3, 1, 3, 1, 4 ])] := [ 1, 3, 1, 3,
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cst[prodim(1, ts, [ 1, 2, 1, 2, 1, 3 ])] := [ 1, 2, 1, 2,
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cst[prodim(1, ts, [ 1, 4, 1, 4, 1, 2 ])] := [ 1, 4, 1, 4,
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cst[prodim(1, ts, [ 4, 1, 4, 1, 3, 4 ])] := [ 4, 1, 4, 1,
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cst[prodim(1, ts, [ 1, 3, 1, 2, 3, 2 ])] := [ 1, 3, 1, 2,
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```
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cst[prodim(1, ts, [ 4, 3, 4, 1, 3, 2 ])] := [ 4, 3, 4, 1,
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cst[prodim(1, ts, [ 4, 3, 4, 2, 3, 1 ])] := [ 4, 3, 4, 2,
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```
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cst[prodim(1, ts, [ 4, 1, 4, 1, 4, 1, 2 ])] := [ 4, 1, 4,
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cst[prodim(1, ts, [ 1, 2, 1, 2, 1, 2, 4 ])] := [ 1, 2, 1,
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cst[prodim(1, ts, [ 1, 3, 1, 3, 1, 3, 2 ])] := [ 1, 3, 1,
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cst[prodim(1, ts, [ 4, 1, 4, 1, 4, 1, 3 ])] := [ 4, 1, 4,
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```

```
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cst[prodim(1, ts, [ 3, 4, 3, 4, 3, 4, 1 ])] := [ 3, 4, 3,
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cst[prodim(1, ts, [ 4, 1, 4, 1, 3, 2 ])] := [ 4, 1, 4, 1,
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cst[prodim(1, ts, [ 1, 2, 1, 2, 3, 4 ])] := [ 1, 2, 1, 2,
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```

```
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cst[prodim(1, ts, [ 1, 4, 1, 4, 1, 4, 1 ])] := [ 1, 4, 1,
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```

```
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cst[prodim(1, ts, [ 4, 1, 4, 1, 4, 1, 4, 1 ])] := [ 4, 1,
4, 1, 4, 1, 4, 1 ];
for i in [1..408] do print i, cst[i]; end for;
G<x,y,t>:=Group<x,y,t|x^3,y^3,(x*y)^2,t^2, (t,x),(y*t)^18 ,
(x*y*t)^16 ,(y*t* (t^(y^2))^x)^4 , (x*y*t*t^(y^2))^9>;
f,G1,k:=CosetAction(G,sub<G|x,y>);
t0:=f(t);
t1:=f(t^y);
t2:=f(t^(y^2));
t3:=f((t^(y^2))^x);
print t0, t1, t2, t3;
print f(x);
print f(y);
```

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